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SOME PROBLEMS IN THE THEORY OF OPTICAL DIFFRACTION

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ABSTRACT. The diffraction of waves by a perfectly reflecting wedge is now a problem of considerable celebrity.

A semi-infinite plane may be looked upon as a wedge whose internal angle is zero. The application of Kirchhoff's formula in the latter case is considerably extended, it being shown that, through a region of not too small dimensions comprising the whole of the boundary of the geometrical shadow, the solution thus obtained is identical with the accurate solution. An expression is found for a plane wave incident at any angle upon a semi-infinite screen.

The diffraction of waves by a straight slit in a plane screen is a problem not yet exactly solved; it is one which occupied Lord Rayleigh's mind for many years. An approximate expression is found for the disturbance distributed over the slit due to a wave incident normally, and by an application of Kirchhoff's formula an important result is obtained very readily, which is in agreement with Rayleigh's investigation by very long analysis.

This appears to be the first time that an attempt has been made to apply Kirchhoff's formula to an area of disturbance which is not considered to be uniform over the area.

The paper gives a brief treatment of the foregoing points of novelty and interest in these two problems which have long been and still are of some importance in mathematical physics.

INTRODUCTION

FEW single problems in the application of mathematics to physics have excited more interest than that of the diffraction of light by a semi-infinite plane. The problem was first exactly solved by Poincaré* for a converging beam and, as he showed later, the case of plane waves, treated by Sommerfeld, is immediately deducible from it. The exact solution in the form of real integrals, when the source is at a finite distance, is given by Macdonald†. Other solutions have been given by

* Poincaré, *Acta Math.* 16 (1892–3) 297.

† Macdonald, *Proc. Lond. Math. Soc.* Ser. 2, 14 (1915) 410.

Macdonald* and Jackson†, of which the latter's seems to be the simpler. The most general and elegant solution, by an extension of the method of images, is due to Bromwich‡. It is the only problem in the theory of diffraction that has been completely solved, thus marking a departure from the older kinematical methods of Fresnel and Kirchhoff and the dynamical theory of Stokes. The importance of the problem is sufficient reason for this communication, for there are several points of interest that appear to be new and worth a brief consideration. In the first place the application of Kirchhoff's formula can be made more rigorous than is usually done. Secondly, a solution can be found for a plane wave incident at any angle. Finally, the foregoing analysis leads very simply to an approximate solution of the problem of the straight slit. It is unnecessary to make any further reference to the contents of the paper in this place, but it may be remarked that it has been possible to treat all the above with the aid of comparatively simple mathematical analysis.

Kirchhoff's Formula

This formula may be considered as, under certain circumstances, the mathematical expression of Huygens' Principle and, as is well known, it leads to the law of the secondary disturbance obtained by Stokes in his great paper upon the Dynamical Theory of Diffraction. A brief consideration of the formula and its application to the problem of the straight edge is unavoidable.

If the surface of resolution be plane§, the formula may be written

$$2\pi\phi_P(t) = \iint \frac{\partial}{\partial n}\left[\frac{1}{r}\phi\left(t-\frac{r}{c}\right)\right] dA \qquad \ldots.(1),$$

where dA is an element of area of the surface. In application the plane of resolution is taken as unaffected by diffraction.

Let the source be situated at S (Fig. 1), and let P be the point at which the effect is to be determined. The section of a plane screen, which is perpendicular to the plane of the paper, is represented by AOB, and the edge of an aperture perforating the screen exists at O. Let OB be the plane of resolution and, taking O as origin, OB as the axis of x, and the perpendicular to the plane of the paper as the axis of y, let the co-ordinates of a point Q be $x, y, 0$. Let the distances of S and P from Q be respectively R_0 and R, and let QS and QP make with the inward-drawn normal to the plane of resolution angles whose complements are ϕ_0 and ϕ. Let r_0, r, θ_0, θ be the corresponding co-ordinates for OS and OP. Let the source at S be represented by $Be^{i\kappa ct}$, where $\kappa = 2\pi/\lambda$ and λ is the wave-length of the emitted disturbance. Assuming that κ is large, it can be shown that the most important term in (1) is given by

$$2\pi\phi_P(t) = i\kappa e^{i\kappa ct}\iint \frac{B}{R_0 R}(\sin\phi - \sin\phi_0)\, e^{-i\kappa(R_0+R)}\, dA \qquad \ldots.(2).$$

* Macdonald, *Electrical Waves.*
† Jackson, *Proc. Lond. Math. Soc.* Ser. 2, 1 (1904) 393.
‡ Bromwich, *ibid.* 14 (1915) 450.
§ Schuster, *Phil. Mag.* 5 (1894) 543.

A short digression may be made here to consider the electromagnetic analogue of Stokes' elastic solid solution. For, writing

$$\pi = C\frac{e^{-ikR}}{R}$$

and f, g, h for the components of electric polarization at P, it is known that

$$f = -\frac{\partial^2\pi}{\partial x\,\partial z},\quad g = -\frac{\partial^2\pi}{\partial y\,\partial z},\quad h = \frac{\partial^2\pi}{\partial x^2} + \frac{\partial^2\pi}{\partial y^2}.$$

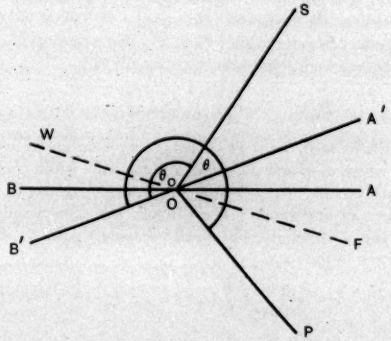

Fig. 1

If the electric force in the wave-front at Q be parallel to the axis of x, and if α be the angle between this axis and QP, it readily follows that the resultant electric polarization at P is given by

$$\kappa^2 C\frac{e^{-ikR}}{R}\sin\alpha.$$

Hence the complete solution is obtained on multiplying the integrand in (2) by $\kappa^2 \sin\alpha$. This is of importance when considering the application of (2) to a semi-infinite screen, if the polarization in the incident wave is parallel to the edge of the screen. For when the incident wave is plane and its direction lies in a plane perpendicular to the edge of the screen it is obvious that $\sin\alpha$ will always be unity.

If the aperture be small, (2) may be written

$$2\pi\phi_P(t) = ik e^{ikct}\frac{B}{r_0 r}(\sin\theta - \sin\theta_0)\iint e^{-ik(R_0+R)}\,dA \qquad \ldots.(3).$$

This equation is applicable when the linear dimensions of the aperture are small compared with r_0 and r.

The Semi-Infinite Plane

This problem is very exceptional in problems of diffraction.

(*a*) Its dimensions do not depend upon any quantity.

(*b*) Particularly, none of its dimensions depend upon the wave-length.

(*c*) It lends itself to an important consideration, namely the choice of the surface of resolution.

On account of (*b*) the solution can be expressed as the first term of a series of ascending powers of λ/r, it being shown that the subsequent terms must disappear. With regard to (*c*), it is not by any means apparent that the area of an aperture is the best area to take as the surface of resolution.

It is first necessary to refer to the effect, at a distant point, of a line of sources. Denoting it by $D(\kappa r)$ according to the usual notation,

$$D(\kappa r) = -\left(\frac{\pi}{2i\kappa r}\right)^{\frac{1}{2}} e^{i\kappa r}\left\{\frac{1^2}{1\cdot 8i\kappa r} + \dots\right\} \qquad \dots(4),$$

this being the suitable form for large values of κr. A luminous point will be taken to constitute S, which of course generates a plane wave when S moves to infinity. It will be assumed initially that r_0 is so great that $e^{-i\kappa r_0}/r_0$ may be considered constant over the area of integration. The distance r is also considered to be great compared with the wave-length, but small compared with r_0. The equation (3) may now be written in the form

$$2\pi\phi_P(t) = e^{i\kappa ci}\, i\kappa \left(\frac{\pi}{2i\kappa r}\right)^{\frac{1}{2}} \frac{B}{r_0}(\sin\theta - \sin\theta_0)\, e^{-i\kappa(r_0+r)} \int e^{i\kappa f(x)}\, dx \dots(5),$$

in which

$$f(x) = x(\cos\theta + \cos\theta_0) + \frac{1}{2}\frac{x^2\sin^2\theta}{r} + \dots \qquad \dots(6).$$

In expression (5) the effective width of the area of integration is, as usual, considered small compared with r, and it will be noticed that we have already neglected all terms in λ/r above that of the lowest power. In $f(x)$ powers of x above the second will be neglected, the omission of the higher powers being justified later.

In the problem under consideration up to this point any plane coterminous with the edge of the screen can be taken as the surface of resolution. Now Kirchhoff's theorem requires that the effect at a point outside a closed surface must be zero when the integration is taken over the whole of the surface. In fact, if OB' be the plane of resolution, the effect at P must certainly remain stationary for small variations in the position of OB', the closed surface being a very thin wedge.

Except at points close to the boundary of the geometrical shadow, $f(x)$ may be represented by its first term. Hence in (5) the terms that are affected, at points not close to this boundary, are $(\sin\theta - \sin\theta_0)$ and $(\cos\theta + \cos\theta_0)$. These terms will vary if another plane be taken as the origin of the disturbances, so that we must write $\theta + \alpha$ for θ and $\theta_0 + \alpha$ for θ_0, and then a plane must if possible be found for which these terms will remain stationary when α is varied.

Consider, therefore, the variations of

$$\sin(\theta+\alpha)-\sin(\theta_0+\alpha) \quad \text{and} \quad \cos(\theta+\alpha)+\cos(\theta_0+\alpha)$$

with α. It is readily found that these terms will remain stationary if $\alpha=-(\theta+\theta_0)/2$. Equation (5) becomes, if the area corresponding to this value of α be taken as the surface of resolution,

$$2\pi\phi_P(t)=e^{i\kappa ct}i\kappa\left(\frac{\pi}{2i\kappa r}\right)^{\frac{1}{2}}\frac{B}{r_0}\,2\sin\frac{\theta-\theta_0}{2}\,e^{-i\kappa(r_0+r)}\int e^{\left\{-i\kappa\right\}-x\,2\cos\frac{\theta-\theta_0}{2}+\frac{1}{2}\frac{x^2}{r}\}}\,dx \quad \ldots.(7).$$

In this equation the coefficient of $\frac{1}{2}\frac{x^2}{r}$ in $f(x)$ has been put equal to unity, for this term is negligible except at points close to the boundary of the geometrical shadow. This is justifiable but appears sometimes to be assumed without a reason being given*.

Let OB (Fig. 1) be the section of the semi-infinite screen, and consider the effect at P of the sources produced by the incident wave upon the complete plane of resolution $A'OB'$ in the absence of the screen. This is of course the same as the effect of the undisturbed incident wave. Hence the actual effect at P is the difference between the undisturbed incident wave and the effect due to the sources distributed over OB' on the supposition that the screen is absent. This latter effect is given by (7).

In accordance with the well-known procedure, based upon the theory of Fresnel's zones, the integral in (7) may be taken from 0 to ∞. For convenience write, in equation (7),

$$\kappa.2\cos\frac{\theta-\theta_0}{2}=l \quad \text{and} \quad \kappa/r=m \qquad \ldots.(8).$$

Then the integral in (7) may be written

$$\int_0^\infty e^{ilx-i\frac{1}{2}mx^2}\,dx \qquad \ldots.(9).$$

Lommel, in his studies of diffraction, was the first to use an integral of this form. It can readily be expressed as a Fresnel integral, in which form it becomes

$$\sqrt{\frac{\pi}{m}}\,e^{i\frac{l^2}{2m}}\int_{-\infty}^{\sqrt{\frac{1}{\pi}\frac{l^2}{m}}}e^{-\frac{1}{2}\pi v^2 i}\,dv \qquad \ldots.(10).$$

When r is infinite outside the boundary of the geometrical shadow, l^2/m becomes infinite. Write therefore for the integral in (10)

$$\left[\frac{e^{i\frac{\pi}{4}}}{\sqrt{2}}\int_{-\infty}^{\sqrt{\frac{1}{\pi}\frac{l^2}{m}}}e^{-\frac{1}{2}\pi v^2 i}\,dv\right]\sqrt{2}\,e^{-i\frac{\pi}{4}} \qquad \ldots.(11),$$

* See Walker, *Analytical Theory of Light.*

so that the term in brackets becomes unity when r becomes infinite. Denote this term by T. It is now possible to write (7) in the form

$$2\pi\phi_P(t) = i\kappa\left(\frac{\pi}{2i\kappa r}\right)^{\frac{1}{2}}\frac{B}{r_0}\,2\sin\frac{\theta-\theta_0}{2}\,e^{-i\kappa r_0}\,e^{-i\kappa(r-ct)}\sqrt{\frac{2\pi}{m}}\,e^{-i\frac{\pi}{4}}\,e^{i\frac{l^2}{m}}\,T \quad \ldots.(12).$$

It will be assumed that r_0 becomes infinite in such a way that

$$\kappa r_0 = 2n\pi,$$

where n is as great an integer as we please, while B/r_0 remains finite and equal to B, say. Hence (12) reduces to

$$\phi_P(t) = B\sin\frac{\theta-\theta_0}{2}\,e^{i\kappa ct - i\kappa r + il^2/2m}\,T.$$

Write I for the incident wave. Then, since

$$l^2/2m = 2\kappa r\cos^2\frac{\theta-\theta_0}{2},$$

$$\phi_P(t) = IT\sin\frac{\theta-\theta_0}{2}.$$

Now, at points not too remote from the boundary of the geometrical shadow, the factor $\sin(\theta-\theta_0)/2$ may, so far as the important features of the phenomena are concerned, be suppressed. Hence at these points the formula may be written

$$\phi_P(t) = IT.$$

The suppressed factor does not affect the position of the diffraction bands, and only very slightly the intensity for a considerable distance on either side of the boundary of the geometrical shadow.

Writing, therefore, ϕ for the total effect at P,

$$\begin{aligned}\phi &= I(1-T)\\ &= I\frac{e^{i\frac{\pi}{4}}}{\sqrt{2}}\int_{-\infty}^{\sqrt{\frac{4\kappa r}{\pi}}\cos\frac{\theta-\theta_0}{2}} e^{-\frac{1}{2}\pi v^2 i}\,dv \qquad \ldots.(13),\end{aligned}$$

in which $\cos\dfrac{\theta-\theta_0}{2}$ is considered to be positive, i.e. $\theta-\theta_0<\pi$. The formula (13) makes the incident wave vanish at infinity within the shadow, for there $\theta-\theta_0>\pi$, while it reproduces the incident wave at infinity outside the shadow. It should perhaps be added that the argument depends upon the justification for taking the amplitude terms outside the sign of integration. This is of course implied in the neglect of powers of λ/r above the first in these terms.

In obtaining the formula (12) it has of course to be assumed that λ is small compared with the distance r of the point P. The extension of the formula to all points rests upon the fact that none of the dimensions of the screen depends upon the wave-length. For, if the solution found is to deal with all wave-lengths, it is applicable for all values of r greater than r_1, so long as a value of λ can be assigned which is such that λ/r_1 can be made to approach ϵ, where ϵ is as small a quantity as we are pleased to choose.

It is necessary, finally, to justify the neglect of those terms involving powers of x in $f(x)$ higher than the second. In order to obtain a solution at every point of space the first and second powers must at least be retained. The second are necessary at points near the boundary of the geometrical shadow. That they are sufficient is easily seen, for the effect of the terms containing x^3 and above becomes vanishingly small compared with that of the previous terms since, by reducing the wave-length, the effective area over which the integration takes place can be made as small as we please.

It will of course be noted that (13) is the exact solution for a semi-infinite plane at all points, for it is periodic with a period 4π. Its complete expression is

$$\phi(t) = Be^{ik(ct+r\cos\theta-\theta_0)} \frac{e^{i\frac{\pi}{4}}}{\sqrt{2}} \int_{-\infty}^{\sqrt{\frac{kr}{\pi}}\cos\frac{\theta-\theta_0}{2}} e^{-\frac{1}{2}\pi v^2 i}\, dv \qquad \ldots.(14).$$

Further, it can be easily verified that (14) satisfies the wave equation. It is convenient to measure θ and θ_0 from the side of the screen towards the incident light. This does not alter (14) and will now be assumed.

The Reflected Wave

The foregoing solution applies to an absorbing screen. If the screen is perfectly reflecting the effect of the reflected wave must be considered. The reflected wave is due virtually to a source $-Be^{ikct}$ at I, where I is the image of S (Fig. 1) in the plane AOB. If this wave be completed, it is equivalent to providing sources

$$-\frac{1}{r_0} Be^{ikct-ikr_0}$$

all along AOB. The effect of these sources along OA must be annulled by supplying sources

$$+\frac{1}{r_0} Be^{ikct-ikr_0}$$

along OA.

Now, if the reflected wave be completed, there is evidently no diffraction. Hence the latter is due to the annulling sources along OA. Diffraction, therefore, due to the reflected wave is the same as though the screen now coincided with OA and the wave proceeded from I. In this case there is no change in θ_0, but $2\pi - \theta$ must be written for θ in (7). Thus $\sin\frac{\theta-\theta_0}{2}$ becomes $-\sin\frac{\theta+\theta_0}{2}$ and $\cos\frac{\theta-\theta_0}{2}$ becomes $-\cos\frac{\theta+\theta_0}{2}$.

Remembering that finally θ and θ_0 were measured from the side of the screen towards the incident light, it is necessary to write, for the reflected wave, in (14) $\theta+\theta_0$ in place of $\theta-\theta_0$, and, since

$$-\sin\frac{\theta+\theta_0}{2}$$

remains unchanged, to prefix a minus sign to the whole expression. Thus in the

case of perfect reflection the complete expression for $\phi\,(t)$ is obtained by subtracting from the right-hand side of (14) a precisely similar expression, but with $\theta + \theta_0$ in place of $\theta - \theta_0$.

It is obviously possible to generalize this solution. For the wave equation in three dimensions is

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} + \kappa^2\phi = 0.$$

Let $\phi = P.Q$, where P is a function of r and θ only, and Q a function of z only. Then the solution applicable, for instance, to a wave travelling in any direction can be found. For example,

$$\phi = Ae^{i\gamma}\int_{-\infty}^{\alpha} e^{-\frac{1}{2}\pi v^2 i}\,dv,$$

where

$$\alpha = \sqrt{\frac{4\,(\kappa^2 - m^2)^{\frac{1}{2}}\,r}{\pi}}\cos\frac{\theta - \theta_0}{2} \quad \text{and} \quad \gamma = mz + (\kappa^2 - m^2)^{\frac{1}{2}}.r\cos(\theta - \theta_0),$$

is a solution. For m may be written $\beta\kappa$, so that β and θ_0 are sufficient to determine the direction of the incident wave.

A solution of the wave equation which makes $\partial\phi/\partial n$ vanish over both sides of the screen is easily shown to be given by adding to the right-hand side of (14) a precisely similar expression, but with $\theta + \theta_0$ in place of $\theta - \theta_0$.

The Problem of a Slit in a Perfectly Reflecting Screen

With a view to investigating the passage of waves through a straight slit in a plane screen, a further consideration of the single straight edge is required. It is assumed that primary waves are incident perpendicularly upon the screen, and it is required to find the value of ϕ over the plane prolongation of the screen OA (Fig. 1). Omitting the time factor, it is found that

$$\left.\begin{aligned}\phi &= B\,\frac{e^{i\frac{\pi}{4}}}{\sqrt{2}}.2\int_0^{\sqrt{\frac{2\kappa r}{\pi}}} e^{-\frac{1}{2}\pi v^2 i}\,dv \\ &= B\int_0^{\sqrt{\frac{2\kappa r}{\pi}}} (1+i)\left(\cos\frac{\pi v^2}{2} - i\sin\frac{\pi v^2}{2}\right)dv \\ &= B\,[(C+S) + i\,(C-S)]\end{aligned}\right\} \qquad \ldots.(15).$$

The square of the amplitude is therefore given by $2B^2\,(C^2 + S^2)$ and the phase by

$$\tan^{-1}\frac{C-S}{C+S}.$$

In Fig. 2, the amplitude is plotted against r, the distance from the edge of the screen. The line AB represents the amplitude of the undisturbed wave. If the polarization is perpendicular to the edge of the screen, then along OA (Fig. 1)

$$\phi = B \qquad \ldots.(16).$$

Hence over OA the incident wave is unaffected. Now for small values of r an approximation to (15) is given by

$$\phi = B(1+i)\sqrt{\frac{4r}{\lambda}}$$

and the amplitude corresponding to this value of ϕ is equal to B when $r = \lambda/8$ (see Fig. 2).

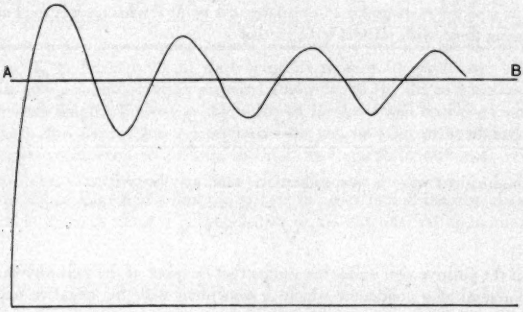

Fig. 2

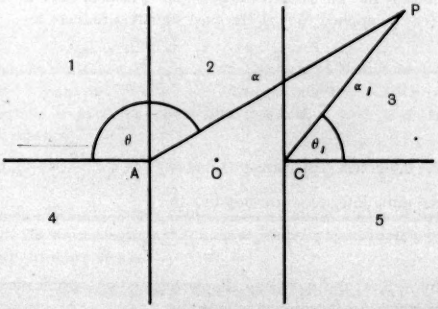

Fig. 3

In Fig. 3, AC represents the section of the slit by a plane perpendicular to its edges. In the first place a plane wave impinges perpendicularly upon the plane of the slit, the polarization of the wave being parallel to an edge of the slit. The width of the slit is supposed to be comparatively great with respect to a wave-length, and will be represented by a. It has been seen that the disturbance produced by diffraction very rapidly falls off as we recede from the edge of the screen, so that,

if the width of the slit be moderately large compared with a wave-length, a very approximate solution is easily obtained, which is applicable at every point to the majority of cases of optical diffraction by a slit.

Consider the function

$$V = \left[e^{imy}\int_{-\infty}^{f} e^{nv^2}\,dv\right]\left[\int_{-\infty}^{f_1} e^{nv^2}\,dv\right] \qquad \ldots.(17)$$

and let x, y be the rectangular co-ordinates of a point P with respect to A as origin, x_1, y_1 being those with respect to C, so that

$$\frac{\partial}{\partial x} = -\frac{\partial}{\partial x_1}.$$

Here f has the form $\pm\sqrt{\frac{4\kappa r}{\pi}}\cos\left(\frac{\theta}{2}\pm\frac{\pi}{4}\right)$

when the incident wave is perpendicular. This may be written

$$\pm\sqrt{\frac{2\kappa}{\pi}(r\pm y)} \qquad \ldots.(18)$$

which, if the positive sign under the square root be taken, is the well-known electrostatic potential of a conductor which is coincident with the negative half of the y axis.

The function f_1 has the same form but with r_1 instead of r. If the differential operator $\nabla^2 + \kappa^2$ be applied to (17), the wave equation reduces to

$$2e^{i\kappa y}\left[-\frac{\partial f}{\partial x}\cdot\frac{\partial f_1}{\partial x_1} + \frac{\partial f}{\partial y}\cdot\frac{\partial f_1}{\partial y_1}\right]e^{-i\frac{\pi}{2}(f^2+f_1^2)} \qquad \ldots.(19),$$

provided that $m = \pm\kappa$ and $n = -i\frac{\pi}{2}$.

Let the first integral in (17), corresponding to $f = \pm\sqrt{\frac{2\kappa}{\pi}(r+y)}$, be denoted by $\pm\alpha$ and let the same integral, corresponding to

$$f = \pm\sqrt{\frac{2\kappa}{\pi}(r-y)},$$

be denoted by $\pm\beta$, α_1 and β_1 having the corresponding significance with respect to f_1. Then an approximate solution is given by

$$\left(\frac{e^{i\frac{\pi}{4}}}{\sqrt{2}}\right)B\left[e^{i\kappa y}\{\alpha\alpha_1 + \beta\beta_1\} - e^{-i\kappa y}\{\beta\alpha_1 + \alpha\beta_1\}\right] \qquad \ldots.(20).$$

This function has the correct form at infinity, for in the spaces 1 and 3 the incident and reflected waves are superposed. In space 2 the incident wave alone exists and in 4 and 5 the function is zero. Furthermore (20) is, together with its first derivatives, continuous at all points and exactly satisfies the boundary conditions.

Now, over the area of the slit, f and f_1 in β and β_1 are negative and y is zero. Hence, over the area of the slit, (20) may be written

$$\frac{(1+i)^2}{4} B(\alpha-\beta)(\alpha_1-\beta_1)$$

or
$$\phi = 2iB\int_0^{\sqrt{\frac{2\kappa x}{\pi}}} e^{-\frac{1}{2}\pi v^2 i}\,dv \int_0^{\sqrt{\frac{2\kappa(a-x)}{\pi}}} e^{-\frac{1}{2}\pi v^2 i}\,dv \qquad \ldots.(21).$$

If the electric force vector in the incident wave be perpendicular to the edge of the screen, then, to the same approximation, provided the width of the slit be not too small, $\phi = B$, the intensity of the wave being the same in the two cases. The foregoing is a very accurate solution provided the width of the slit be large compared with the wave-length. So long as the width of the slit is well over one-eighth of a wave-length, the above solution is approximate, and its main point of interest lies in its simple verification of a result obtained by Lord Rayleigh* after very long analysis.

Let us apply Kirchhoff's formula to (21) in order to find the disturbance at a considerable distance along a perpendicular to the slit. A first approximation is given by

$$\phi = 2iB\frac{2\kappa}{\pi}\sqrt{x(a-x)} \qquad \ldots.(22).$$

The effect required is proportional to the integral of ϕ taken over the slit, i.e. to

$$4iB\frac{\kappa}{\pi}\int_0^a \sqrt{x(a-x)}\,dx = \frac{iB\kappa a^2}{2} \qquad \ldots.(23).$$

In order to find the width of a slit which just allows a disturbance whose polarization is parallel to the edge to pass through with the same facility as it allows a disturbance whose polarization is perpendicular to the edge, it is only necessary to equate $B\kappa a^2/2$ to Ba, whence

$$a = \frac{\lambda}{\pi}.$$

Lord Rayleigh had previously shown that, when the width of the slit is small compared with the wave-length, a disturbance, whose polarization is parallel to the edge, is almost entirely reflected.

* Rayleigh, *Proc. Roy. Soc.* A. 89 (1913) 194.

DISCUSSION

Mr J. Guild: Mr Hanson has explained how his method of treatment gives the result that for very small slits the transmitted light is polarized perpendicularly to the length of the slit, as shown by the late Lord Rayleigh. Does Rayleigh's other result, namely, that for slits wider than a certain critical value the polarization is parallel to the slit, also follow from the treatment?

Dr R. S. Clay: Is the effect on the amplitude near the edge of the surface of resolution independent of the material of which the edge is constructed?

Mr T. Smith: Everyone who has attempted to make investigations on diffraction problems will appreciate the great difficulty of the problem with which Mr Hanson is dealing. Any new results in this field are likely to be valuable. Without a fuller opportunity of examining the paper, discussion is difficult. I would like to take this opportunity of drawing attention to a feature of some treatments of diffraction problems which seems to call for fuller explanation. In the integrals at some stage a function is expanded, and it is pointed out that at all points of the surface over which the integral has to be evaluated the leading term or terms are large in comparison with the remainder. The integrals involving only these larger terms are considered, and found to be finite or in certain cases zero. It is concluded that the smaller terms may safely be neglected. This argument would be sound were the integrand everywhere of the same sign, but in the problems of diffraction the signs are frequently changing. The particular cases in which the leading integrals yield a zero result illustrate this change of sign. Discussions which rest on such an assumption appear to me quite inconclusive.

The difficulties of the problems of greatest practical interest are such that their exact solution is not likely to be attained soon. It is not so obvious that precise experimental investigation of the phenomena is also beset with great difficulties. Visual examination offers some advantages over photographic methods, but precision involves the comparison of parts of patterns in close contact. Such patterns may be arranged in juxtaposition in space under conditions which preclude any mutual disturbance, but since the visual impressions depend on the images formed on the retina it appears almost impossible to realize the conditions required for very sensitive critical experiments.

Mr Hanson: Mr Smith has called attention to a very important feature of the expansions employed in certain physical problems. The approximation adopted in forming equation (22) is certainly only valid for slits of very small width, and it is admitted that further consideration is necessary before it is possible to obtain an answer to Mr Guild's question. Fresnel's integrals were developed by Knochenhauer in the form of the sum of two convergent series. These series are suitable for calculation when the upper limit is small. Cauchy likewise developed the integrals in series which are suitable for calculation when the upper limit is large.

Neither of these series is, unfortunately, applicable in the present case and resort has therefore been had to numerical calculation.

The expression on the right-hand side of equation (21) has been evaluated with the aid of Gilbert's Table of Fresnel's Integrals for slits whose widths are $\cdot 3\lambda$, $\cdot 4\lambda$, $\cdot 5\lambda$, $\cdot 7\lambda$, and λ respectively. Let a beam of unpolarized light fall normally upon the slit, the vibrations being respectively parallel to and perpendicular to the edges of the slit. Let the ratio of the intensity of the component of the transmitted beam which is polarized in a direction parallel to the slit to the intensity of that which is polarized in a direction which is perpendicular to the slit be ρ. Then ρ has the following values:

a/λ	ρ
·3	·19
·4	1·12
·5	1·46
·7	1·44
1·0	·96

Since the calculations are somewhat laborious, only these five cases have been evaluated. They are, however, sufficient to show that the value λ/π, obtained above, must be slightly increased. They also show that, whereas for slits of very small width the predominant polarization of the transmitted beam is perpendicular to the edges of the slit, for slits of greater width the polarization of the transmitted beam is reversed. It is further probable that, for slits whose widths are greater than λ, the ratio does not differ appreciably from unity. It is noteworthy that the ratio increases very rapidly between the values $\cdot 3$ and $\cdot 4$ of a/λ.

In answer to Dr Clay's question it should be stated that the solution of the problem, when the optical properties of the material are arbitrary, has not been attempted. The case considered is that of a screen which satisfies the ideal condition of being a perfect reflector. If the screen be perfectly opaque, so that all incident radiation is absorbed, the amplitude near the edge is not the same as in the previous case. It may be concluded from these two ideal cases that the amplitude near the edge is not independent of the material of which the edge is composed.

THE ANNEALING AND RE-ANNEALING OF GLASS

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ABSTRACT. The factors which play an important part in the annealing of glass are discussed. Relations connecting the strains present when cooling commences, strain due to gradient, and strain present after cooling are obtained and the equations are applied to the case of slabs. Tables showing the proportion of temporary strain remaining as permanent strain for different kinds and sizes of slabs are given. The considerations affecting the temperature at which annealing is carried out and the conditions for the total annealing and cooling times to be a minimum are then dealt with, the results being applied to a particular case. The effect of errors in pyrometry are discussed and tables for annealing various kinds of glass are given.

PART I. THE ANNEALING OF GLASS

Introduction

If a piece of glass be heated or cooled, there must be a temperature gradient across it during the operation; otherwise heat could not flow in or out of it. Considering, first, glass whose temperature is changing but which is all at a temperature outside the range at which viscous flow can occur, i.e. in practice below about 300° C., this temperature gradient calls for a change of length in the outer layers relative to the inner layers. This necessitates strains in the material and corresponding stresses. These strains only exist, however, while the temperature gradient exists, as the glass is elastic at such temperatures. If, therefore, a piece of glass initially free from strain be heated, certain stresses appear in it, compression in the outer layers and tension in the inner, these stresses being related to the rate of heating by formulae discussed later.

These strains and stresses are called *temporary*, as they disappear when the gradient is removed. The same remarks apply to the case of glass being cooled, the signs of the stresses being reversed. When, however, the glass is heated to a sufficiently high temperature, the strain is found to disappear even though the gradient be maintained. This is caused by the actual flow of the material of the block under the stress caused by the gradient. Consider a block of glass at a sufficiently high temperature for all strain to disappear in a few moments. The glass is now cooled at some definite rate, causing a temperature gradient across it. The resultant strains are immediately relieved by plastic flow and as long as the rate of cooling is not changed no strain will

be present even when the temperature has fallen sufficiently for the glass to be elastic. This is easily demonstrated by cooling a block of glass under conditions where the double refraction—as strain in glass is always measured in terms of the double refraction caused by it—can be viewed. At some lower temperature—say room temperature—the temperature gradient is removed and then *reverse* strains appear which are exactly equal but opposite in sign to those lost at the high temperature. This is the permanent strain which it is the object of annealing to avoid.

Influence of Temperature on Rate of Disappearance of Strain

It is well known that strain is relieved at an increasing rate as the temperature of the glass is raised. At temperatures below 300–400° C. this rate is too low to be measured. Adams and Williamson* have published figures, from which the following are abstracted (Table I), giving the increase in temperature necessary to double the rate of release of strain.

Table I. *Data abstracted from Adams and Williamson's paper*

Type of glass	Optical properties		Rate doubled each rise of °C.	θ_0 °C.
	n_D	ν		
B.S.C.	1·516	62	10	530
H.C.	1·523	59	10	503
L.B.C.	1·574	57	9	543
D.B.C.	1·608	57	8	575
B.F.	1·606	44	11	480
L.F.	1·573	42	10	400
M.F.	1·616	37	8	408
H.F.	1·655	33	8	395
E.H.F.	1·756	27	9	370

If λ = velocity constant for the relief of strain at any temperature

and d = temperature interval in which λ is doubled,

then
$$\lambda = 2^{\frac{\theta-\theta_0}{d}} \qquad \ldots.(1),$$

where θ = temperature of glass

and θ_0 = value of θ when $\lambda = 1$.

Rate of Release of Strain

Twyman† assumed Clerk Maxwell's suggestion‡ that the rate of release might be proportional to the strain present at any time, that is,

$$\frac{ds_t}{dt} = -bs_t \qquad \ldots.(2)$$

or
$$s_t = s_{t_0} e^{-bt} \qquad \ldots.(3),$$

where s_{t_0} = strain when $t = 0$;

s_t = strain at any time t;

b = constant;

e = base of natural logarithms.

* Adams and Williamson, *Journ. Franklin Inst.* 190 (1920) 597, 835.

† Twyman, *Journ. Soc. Glass Tech.* 1 (1917) 61.

‡ Clerk Maxwell, *Trans. Roy. Soc. Edin.* 20 (1853) 87; *Scientific Papers*, 1, 30.

This does not agree well with the observed facts and Adams and Williamson found experimentally that the law

$$\frac{ds_t}{dt} = -\lambda s_t^2 \qquad \ldots\ldots(4)$$

or

$$\frac{1}{s_t} - \frac{1}{s_{t_0}} = \lambda t \qquad \ldots\ldots(5)$$

is a much closer approximation, where the symbols have the meanings given above.

The Meaning and Value of θ_0

As stated above, strain is measured in terms of the double refraction caused by it, i.e. in terms of wave-lengths of light (assumed 5500 A.U.). θ_0 is then defined as that temperature at which a strain of ·10 wave-length per cm. is reduced to ·05 wave-length per cm. in 10 minutes. From (5)

$$\frac{1}{\cdot 05} - \frac{1}{\cdot 1} = 10\lambda \text{ or } \lambda = 1.$$

Alternatively θ_0 is that temperature where $\lambda = 1$, as defined above. This temperature then only depends on the kind of glass, and annealing temperatures can be given in terms of θ_0 and are applicable generally to all glasses.

Strain produced by Linear Heating or Cooling

In this paper only *linear* rates are considered and for such rates it is proved later that the temporary strain depends on the size and shape of the glass and is also directly proportional to the rate of heating, that is,

$$s_0 = pc \qquad \ldots\ldots(6),$$

where s_0 = temporary strain due to heating at rate p ° C./min.,

and c = constant depending on the size and shape of the glass.

The permanent strain present in a block of cold glass is of the same sign as the temporary strain induced by heating and this distribution is assumed to be positive. Of course, both tension and compression exist in the same piece, but the distribution changes sign with varying conditions. Thus, on cooling, the temporary strain is opposite in sign to that present in cold glass and is therefore negative.

The following cases now arise during cooling and these must be considered separately.

First, when no viscous flow can occur. If there is no permanent strain originally present, then a strain $-s_0$ exists during cooling which disappears when the gradient is removed and $S = 0$, where S = final strain in cold glass. If a strain $+s_a$ is present when cooling is commenced and strain $-s_0$ is added due to cooling, then $s_t = s_a - s_0$, the sign of s_t depending upon the relative magnitudes of s_a and s_0. When the glass is cold $-s_0$ disappears and $S = s_a$.

When strain is lost by viscous flow the sign of the strain lost depends on the sign of $(s_a - s_0)$, being negative when this is negative. When s_0 is numerically

greater than s_a, s, i.e. the strain lost by plastic flow, is negative and reappears as positive when the glass is cold. Thus the final strain $S = s_a + s$ and the strain during cooling

$$-s_t - s = s_a - s_0$$

or

$$-s_t = s_a - s_0 + s \quad \text{(Case I)} \qquad \ldots.(7).$$

When s_a is greater than s_0, s is positive and reappears when the glass is cold as negative strain, so that

$$S = s_a - s$$

and

$$s_t = s_a - s_0 - s \quad \text{(Case II)} \qquad \ldots.(8).$$

Case I. In this case s_t is negative, so that from (4)

$$\frac{-ds_t}{dt} = -\lambda(-s_t)^2.$$

Hence

$$-ds = -\lambda(s_a - s_0 + s)^2\,dt \text{ from (7).}$$

But $S = s_a + s$ (see above), and therefore

$$\frac{-dS}{dt} = -\lambda(S - s_0)^2.$$

If cooling commences at $\theta_0 + n$ °C., the temperature θ at any time t is given by $\theta_0 + n - pt$, where p = rate of cooling in degrees C. per minute.

Hence

$$\lambda = 2^{\frac{n-pt}{d}} \text{ from (1)}$$

and

$$\frac{dS}{(S - s_0)^2} = 2^{\frac{n-pt}{d}}\,dt.$$

As $S = s_a$ for $t = 0$, this leads to

$$\frac{1}{S - s_0} - \frac{1}{s_a - s_0} = -2^{n/d}\frac{d}{p\log_e 2}\left[1 - 2^{\frac{-pt}{d}}\right] \qquad \ldots.(9).$$

In considering the final strain, when the glass is uniform in temperature and at a low temperature, t becomes large and $2^{\frac{-pt}{d}}$ negligibly small.

Again from (6) $p = \frac{s_0}{c}$, so that

$$S = s_0\left[\frac{s_a + (s_0 - s_a)\dfrac{cd\,2^{n/d}}{\log_e 2}}{s_0 + (s_0 - s_a)\dfrac{cd\,2^{n/d}}{\log_e 2}}\right] \qquad \ldots.(10).$$

Case II. Similar reasoning leads to the result

$$S = s_0\left[\frac{s_a + (s_a - s_0)\dfrac{cd\,2^{n/d}}{\log_e 2}}{s_0 + (s_a - s_0)\dfrac{cd\,2^{n/d}}{\log_e 2}}\right] \qquad \ldots.(11).$$

These equations give the value of the strain S that will appear in the glass when it is cold in terms of the temporary strain s_0 due to the cooling, the initial strain s_a when the cooling was commenced, and the temperature n at which cooling commenced, in degrees C. above θ_0, for any glass for which c of (6) is known.

It should be noticed that both (10) and (11) involve the difference between s_0 and s_a as a positive quantity. It should also be noted that, when $s_a = 0$, equation (10) represents the case since the strain during cooling must in this case be negative. Putting $s_a = 0$ in (10) gives

$$S = s_0 \left[\frac{\frac{cd\, 2^{n/d}}{\log_e 2}}{1 + \frac{cd\, 2^{n/d}}{\log_e 2}} \right] \qquad \ldots\ldots(12),$$

which covers the case of glass cooled from a relatively high temperature when strain is immediately removed—hence $s_a = 0$. This also entails a large value for n which makes $2^{n/d}$ large compared with 1. In other words, the portion in the bracket rapidly approaches the value 1, or for cooling from very high temperatures

$$S = s_0.$$

That is, all the strain due to the gradient will appear in the cold glass. This point is indicated again in Tables III and IV at the end of this part of the paper.

Evaluation of the Constant c.

Williamson* gives various equations for the stress induced by linear heating rates in glass of various geometrical shapes. Thus for a large flat plate the stress F parallel to the surface is given as

$$F = \frac{ap}{6K(e-f)}(b^2 - 3x^2) \qquad \ldots\ldots(13),$$

where $(e-f)$ is equal to $\frac{1-\sigma}{E}$,

σ = Poisson's ratio;
E = Young's modulus;
F = stress in kg./cm.^{2};
a = coefficient of linear expansion;
b = half the thickness of the plate;
x = distance from the centre of the plate;
K = diffusivity = $\frac{\text{thermal conductivity}}{\text{specific heat} \times \text{density}}$.

Values of the various constants are given for the series of glasses quoted in Table I and in Table II values of $a/6K(e-f)$ and of the stress-optical coefficient are given. This latter constant is the double refraction per cm. length caused by a

* Williamson, *Journ. Wash. Acad. Sciences*, 9 (1919) 216.

stress of 1 kg./cm.^{2} and the product of the two sets of figures gives the constant D for a plate in the equation

$$F = D\,p\,(b^2 - 3x^2) \qquad \ldots.(14),$$

where F is now measured in terms of double refraction and is therefore the optical strain due to the gradient, i.e. s_0.

Table II

Type of glass	$\frac{a}{6K\,(e-f)}$	Stress-optical coefficient	D
B.S.C.	4·3	·0052	·022
H.C.	4·9	·0047	·023
L.B.C.	5·6	·0051	·029
D.B.C.	4·4	·0039	·017
B.F.	4·7	·0057	·027
L.F.	4·3	·0058	·025
M.F.	4·1	·0057	·023
H.F.	4·4	·0048	·021
E.H.F.	4·7	·0022	·010

Now strain is conveniently measured at the centre of the plate when viewed edgewise, i.e. where $x = 0$,

$$s_0 = Dpb^2 \qquad \ldots.(15)$$

and

$$c = Db^2 \qquad \ldots.(16)$$

for a flat plate.

It should be noticed that the evaluation of this constant assumes that the temperature gradient is removed at ordinary room temperature and would not give the correct value of s_0 at, say, 300° C. Substitution of the value of c may now be made in equations (10), (11), and (12), and in the following tables figures are worked out for the values of the ratio $\frac{S}{s_0}$ from (12). This gives a measure of the proportion of the temporary strain due to the gradient which becomes permanent when cooling is commenced from different temperatures. It is of interest to notice that the initial temperature must be lowered $2d$ ° C. for each doubling of the thickness in order that the ratio $\frac{S}{s_0}$ shall remain constant.

Thus from (12)

$$\frac{S}{s_0} = \frac{\dfrac{Db^2 d}{\log_e 2}\,2^{n/d}}{1 + \dfrac{Db^2 d}{\log_e 2}\,2^{n/d}}$$

for a plate of thickness $2b$. If the thickness be $4b$, then

$$\frac{S}{s_0} = \frac{\dfrac{4Db^2 d}{\log_e 2}\,2^{n'/d}}{1 + \dfrac{4Db^2 d}{\log_e 2}\,2^{n'/d}}$$

where n' = new annealing temperature.

Equating the right-hand sides leads to the result that

$$2^{n/d} = 4 \cdot 2^{n'/d} = 2^{\frac{n'+2d}{d}}$$

or

$$n - n' = 2d.$$

Table III A

Value of ratio S/s_0 for $d=8$, $D=\cdot025$.
This covers approximately the case of M.F. and H.F.

$$\frac{S}{s_0} = \frac{\cdot29b^2 2^{n/8}}{1+\cdot29b^2 2^{n/8}}.$$

Thickness of plate in cm.	·5	1	2	4	8
Value of b in cm.	·25	·5	1	2	4
Value of n in ° C. above θ_0:					
50	·585	·850	·958	·988	·996
40	·366	·700	·905	·974	·994
30	·196	·485	·790	·940	·984
20	·092	·288	·618	·866	·963
10	·043	·146	·407	·730	·918
0	·018	·068	·222	·535	·821
− 10	·007	·029	·120	·325	·660
− 20	·003	·012	·051	·171	·454
− 30	·001	·005	·022	·078	·254
− 40	—	·002	·009	·034	·125
− 50	—	·001	·004	·015	·056
− 60	—	—	·002	·006	·025
− 70	—	—	·001	·003	·011
− 80	—	—	—	·001	·005
− 90	—	—	—	—	·002
− 100	—	—	—	—	·001

Table III B

Value of ratio S/s_0 for $d=8$, $D=\cdot017$.
Case of D.B.C.

$$\frac{S}{s_0} = \frac{\cdot20b^2 2^{n/8}}{1+\cdot20b^2 2^{n/8}}.$$

Thickness of plate in cm.	·5	1	2	4	8
Value of b in cm.	·25	·5	1	2	4
Value of n in ° C. above θ_0:					
50	·488	·792	·940	·985	·997
40	·281	·613	·863	·963	·992
30	·145	·406	·730	·918	·978
20	·065	·218	·525	·820	·950
10	·029	·107	·325	·666	·885
0	·012	·048	·167	·445	·763
− 10	·005	·021	·074	·254	·575
− 20	·002	·009	·035	·123	·368
− 30	·001	·004	·015	·056	·194
− 40	—	·002	·006	·025	·091
− 50	—	·001	·003	·010	·040
− 60	—	—	·001	·004	·016
− 70	—	—	—	·002	·008
− 80	—	—	—	·001	·003
− 90	—	—	—	—	·001
− 100	—	—	—	—	—

Table III c

Values of S/s_0 for $d=9$, $D=\cdot029$.
Case of L.B.C.

$$\frac{S}{s_0}=\frac{\cdot38b^2\,2^{n/9}}{1+\cdot38b^2\,2^{n/9}}.$$

Thickness of plate in cm.	·5	1	2	4	8
Value of b in cm.	·25	·5	1	2	4
Value of n in ° C. above θ_0:					
50	·525	·816	·948	·985	·996
40	·338	·667	·890	·970	·993
30	·194	·488	·788	·937	·984
20	·099	·306	·638	·875	·966
10	·048	·167	·445	·763	·930
0	·023	·087	·271	·600	·857
− 10	·010	·042	·153	·419	·737
− 20	·005	·018	·074	·243	·563
− 30	·002	·009	·036	·131	·376
− 40	·001	·004	·018	·065	·219
− 50	—	·002	·008	·030	·115
− 60	—	·001	·004	·015	·057
− 70	—	—	·002	·007	·027
− 80	—	—	·001	·003	·013
− 90	—	—	—	·001	·006
−100	—	—	—	—	·003

Table III d

Values of S/s_0 for $d=10$, $D=\cdot025$.
Case of H.C., B.S.C. and L.F.

$$\frac{S}{s_0}=\frac{\cdot36b^2\,2^{n/10}}{1+\cdot36b^2\,2^{n/10}}.$$

Thickness of plate in cm.	·5	1	2	4	8
Value of b in cm.	·25	·5	1	2	4
Value of n in ° C. above θ_0:					
50	·419	·743	·921	·979	·995
40	·265	·591	·852	·960	·989
30	·153	·419	·743	·921	·979
20	·083	·265	·591	·852	·960
10	·043	·153	·419	·743	·921
0	·022	·083	·265	·591	·852
− 10	·010	·043	·153	·419	·743
− 20	·005	·022	·083	·265	·591
− 30	·003	·010	·043	·153	·419
− 40	·001	·005	·022	·083	·265
− 50	—	·003	·010	·043	·153
− 60	—	·001	·005	·022	·083
− 70	—	—	·003	·010	·043
− 80	—	—	·001	·005	·022
− 90	—	—	—	·003	·010
−100	—	—	—	·001	·005

Table IV. *Comparative table of the ratio S/s_0 for various types of glass in slabs 2 cm. thick*

Value of n	M.F., H.F.	D.B.C.	L.B.C.	H.C., B.S.C., L.F.
50	·958	·940	·948	·921
40	·905	·863	·890	·852
30	·790	·730	·788	·743
20	·618	·525	·638	·591
10	·407	·325	·445	·419
0	·222	·167	·271	·265
− 10	·120	·074	·153	·153
− 20	·051	·035	·074	·083
− 30	·022	·015	·036	·043
− 40	·009	·006	·018	·022
− 50	·004	·003	·008	·010
− 60	·002	·001	·004	·005
− 70	·001	—	·002	·003
− 80	—	—	·001	·001
− 90	—	—	—	—
− 100	—	—	—	—

The foregoing, while of theoretical interest, is not of great practical value, as it is impossible to estimate the value of s_a during the cooling of plates from high temperatures. The usual practice is to allow plates to cool at their natural rate down to a predetermined temperature, then to hold them at this temperature for some indefinite period, and next to cool them at a definite rate. As s_a cannot be determined, tables can only be constructed for the value $S_a = 0$ and this is the only practical use of Tables III, namely, to indicate the relative effects of various annealing temperatures. The suitable rate of cooling can also be derived to give S a desired value. Thus from Table III D it appears that, for H.C. plates 2 cm. thick, ·265 of the strain due to the gradient will appear in the cold glass when $s_a = 0$ and cooling starts at $n = 0$. If s is to be, say, ·01 unit, then

$$s_0 = \frac{\cdot 01}{\cdot 265} = \cdot 038.$$

Also from (15)

$$s_0 = Dph^2 = \cdot 025p$$

or

$$p = \frac{\cdot 038}{\cdot 025} = 1 \cdot 5^\circ \text{ C. per minute.}$$

This rate of cooling under the assumed conditions will then result in the final strain being ·01 unit per cm.

PART II. THE RE-ANNEALING OF GLASS

Introduction

The operation of re-annealing is employed to remove strain which exists in plates or articles moulded in glass. This strain may be due to faulty annealing in the first place or to subsequent operations involving the heating of the glass above the softening range.

In the present part of the paper the case is discussed for *linear* heating and cooling with special reference to slabs, use being made of the equations derived in the previous part of the paper. The symbols are the same as those previously specified with additions where necessary.

S_1 indicates the strain present in the cold glass when re-annealing is commenced, measured as before in A.U. per cm. length at the centre of the plate.

Considerations affecting the Temperature at which Annealing is carried out

Re-annealing consists in heating the slab to a predetermined temperature, holding it at this temperature until the strain is reduced to a given value s_a, then cooling at a rate p ° C. per minute, which rate is sufficiently slow to prevent the final strain S exceeding a suitable value, which is taken as ·01 A.U. per cm.* at the centre of the slab. From the previous part of the paper it appears that the time t_a to hold at uniform temperature, i.e. $\theta_0 + n$, where θ_0 has been defined and has a fixed value for a given glass, will be rapidly reduced as n is increased.

From (5), substituting suitable values,

$$t_a = \frac{1}{\lambda_{t_a}}\left(\frac{1}{s_a} - \frac{1}{S_1}\right),$$

where $$\lambda_{t_a} = 2^{\frac{\theta_0+n-\theta_0}{d}} = 2^{n/d} \text{ from (1)},$$

or $$t_a = \frac{1}{2^{n/d}}\left(\frac{1}{s_a} - \frac{1}{S_1}\right) \qquad \ldots.(17).$$

The factor acting against the raising of the temperature is the fact that, as shown in Tables III A, B, C, D and Table IV, the proportion of the strain s_0 due to the gradient which becomes permanent is rapidly increased as the temperature from which cooling is commenced is raised. This means that when cooling from a temperature low compared with θ_0 a higher cooling rate may be employed without increasing the strain in the final condition. It may be assumed that at temperatures below $(\theta_0 - 100)$ no permanent strain will be introduced (see Tables III). Then the cooling time for this temperature range is given by

$$t_c = \frac{\theta_0 + n - \theta_0 + 100}{p} = \frac{n + 100}{p},$$

the rate below the lower limit being governed only by considerations of the tensile strength of the glass, i.e. that s_0 shall not be great enough to cause fracture.

* See Adams and Williamson, *Journ. Franklin Inst.* 190 (1920) 597, 835.

From (15)

$$s_0 = Dpb^2 \quad \text{or} \quad p = \frac{s_0}{Db^2}.$$

Substituting this in the above equation gives

$$t_c = \frac{(n + 100)\, Db^2}{s_0} \qquad \ldots.(18).$$

Conditions for the Total of the Annealing and Cooling Times to be a Minimum

The commercial consideration controlling re-annealing is that the cost shall be a minimum, i.e. that the total time $t_a + t_c$ shall be a minimum for a given value of S (= ·01 in this case). That is,

$$T \text{ (total time)} = \frac{1}{2^{n/d}}\left(\frac{1}{s_a} - \frac{1}{S_1}\right) + \frac{(n + 100)\, Db^2}{s_0}$$

shall be a minimum.

Unfortunately, the equations for the evaluation of s_0, s_a, and n—the three variables—are so complex that differentiation is of little use, but numerical solution leads to useful results.

Elimination of s_0

Considering first the case when the strain during cooling is negative, equation (10) gives

$$s_0 = \frac{-\{s_a - S - c\,(s_a + S)\} + \sqrt{\{s_a - S - c\,(s_a + S)\}^2 - 4s_a\, Sc^2}}{2c} \qquad (19),$$

where

$$c = \frac{Db^2 d\, 2^{n/d}}{\log_e 2}.$$

When s_a is less than S, s_t is negative and this equation (19) is used and, as s_0 must be as large as possible to give the least time for t_c, the positive value of the root is needed.

For Case II, i.e. when s_a is greater than the final strain S,

$$s_0 = \frac{-\{S - s_a - c\,(s_a + S)\} - \sqrt{\{S - s_a - c\,(s_a + S)\}^2 - 4s_a\, Sc^2}}{2c} \qquad (20).$$

The following tables have been prepared by assuming a value for n and then solving equations (19) or (20) for s_0 in terms of assumed values for S and s_a and substituting in equations (17) and (18). S_1 has been assumed in all cases to be ·50, which is as bad as is usually found in unfractured glass slabs.

Slabs of B.S.C., H.C., or L.F. 2 cm. thick

This case has been worked out rather fully to indicate the differences in total time needed with varying values of s_a and n.

As in Table III D D = ·025, d = 10. Therefore $C = \cdot 36 \,.\, 2^{n/10}$.

Table V

	s_a (assumed)	s_0	t_s	t_a	T
n assumed = 0° C					
	·002	·0338	74	498	572
	·004	·0292	86	248	334
	·005	·0272	92	198	290
	·006	·0249	100	165	265
	·007	·0221	113	141	254
	·008	·0196	127	123	250
	·009	·0162	154	109	263
	·010	·0100	250	98	348
	·011	·0061	410	89	499
	·012	·0053	472	81	553
	·013	·0049	510	75	585
n = 5° C.					
	·004	·0242	109	175	284
	·005	·0225	117	140	257
	·006	·0213	123	117	240
	·007	·0194	135	100	235
	·008	·0170	155	87	242
	·009	·0148	178	77	255
	·010	·0104	253	69	322
n = 10° C.					
	·004	·0202	136	124	260
	·005	·0192	143	98	241
	·006	·0180	153	82	235
	·007	·0170	162	70	232
	·008	·0154	179	62	241
	·009	·0137	200	54	254
	·010	·0101	273	49	322
n = 15° C.					
	·004	·0177	162	88	250
	·005	·0170	169	69	238
	·006	·0161	178	58	236
	·007	·0154	187	50	237
	·008	·0142	203	43	246
	·009	·0129	223	38	260
	·010	·0100	287	35	322

These figures are plotted in the figure, which shows the rapid increase in the total time as the distribution of the time between annealing and cooling is varied.

Plotting the minima from the above tables gives the values $n = 9°$ C., $s_a = ·007$ to make the total time a minimum, whence

$$s_0 = ·0174.$$

From (15)

$$s_0 = Dpb^2 \quad \text{or} \quad p = \frac{s_0}{Db^2} = \frac{·0174}{·025} = 7° \text{ C. per minute.}$$

Thus the re-annealing should be carried out for this size of slab at a temperature of $\theta_0 + 9°$ C., the glass held at this temperature for 75 minutes, then cooled to $\theta_0 - 100$ at a rate of ·7° C. per minute. The rate may then be increased without affecting the final strain, care only being necessary to prevent fracture of the glass.

The foregoing assumes, of course, ideal conditions as to accuracy of pyrometry and control of cooling. It is of interest to discuss the effect on the final strain of errors in the temperature $\theta_0 + n$ and of errors in the rate of cooling.

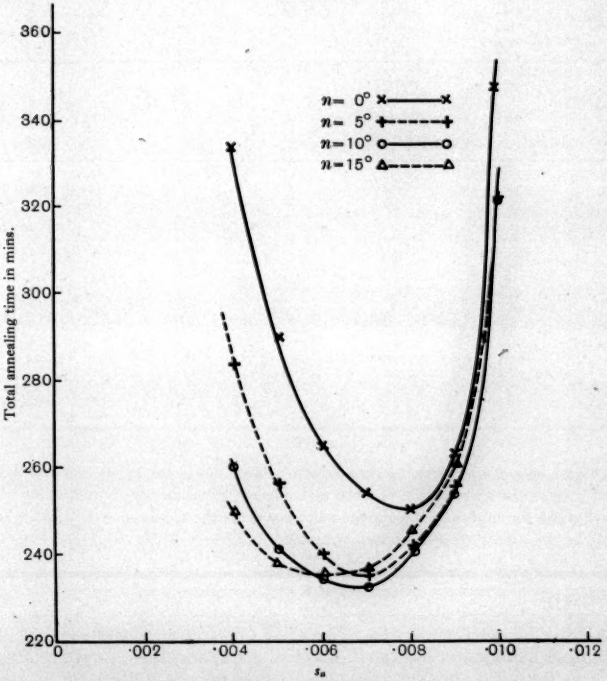
Effect of Errors in $\theta_0 + n$

Assuming that t_a is made 75 minutes but that an error of $-10°$ is made in $\theta_0 + n$, then s_a is given by

$$75 = \frac{1}{2^{-\cdot 1}}\left(\frac{1}{s_a} - \frac{1}{S_1}\right),$$

whence

$$s_a = \cdot 0139.$$



Effect of variations in the values of s_a and n on the total annealing time.

The value of s_0 is ·0174, whence from (10)

$$S = s_0 \left[\frac{s_a + (s_0 - s_a)\dfrac{Db^2d\,2^{n/d}}{\log_e 2}}{s_0 + (s_0 - s_a)\dfrac{Db^2d\,2^{n/d}}{\log_e 2}}\right];$$

$S = \cdot 0141$ instead of $\cdot 0100$.

Similarly an error of + 10° C. in $\theta_0 + n$ leads to a value of

$$S = \cdot 0107 \text{ instead of } \cdot 0100.$$

It is therefore preferable to allow for negative errors in pyrometry for the annealing temperature and, if it be granted that temperatures may be assumed to ± 10° C., then the time must be increased from 75 to 130 minutes to cover this possibility. This error has been allowed for in the tables by doubling the value of t_a given by the equation (17) above.

Errors in the Rate of Cooling

The direct effect of errors in the cooling rate will be to increase or decrease s_0, so that the final value of S will be nearly (not exactly, owing to the effect of s_a) proportionally in error. Thus for the case already considered, if p be doubled,

$$s_0 = \cdot 0348$$

and

$$S = \cdot 0170.$$

Thus for safety it is wise to use half the rates of cooling derived from equation (15) and these values are tabulated, not the actual figures given by (15).

Table VI A

Values of s_a and n for
M.F. and H.F.
to make total re-annealing time a minimum
$D = \cdot 025$ $d = 8$

Thickness of plate in cm. ...	·5	1	2	4	8
Value of b in cm.	·25	·5	1	2	4
s_a	·007	·007	·007	·007	·007
n ° C. above θ_0	+38	+ 23	+ 9	− 5	− 20
t_a in mins.	11	38	129	435	1600
t_s in mins.	21	80	305	1130	3880
Total time in mins. ...	32	118	434	1565	5480
Rate of cooling p = °C. per min.	6·6	1·5	·35	·08	·02

A factor of safety of 2 has been applied to the above figures.

Table VI B

Values of s_a and n for
D.B.C.
to make total re-annealing time a minimum
$D = \cdot 017$ $d = 8$

Thickness of plate in cm. ...	·5	1	2	4	8
Value of b in cm.	·25	·5	1	2	4
s_a	·007	·007	·007	·007	·007
n ° C. above θ_0	+42	+27	+ 11	− 4	− 20
t_a in mins.	8	30	110	400	1600
t_s in mins.	15	55	195	700	2300
Total time in mins.	23	85	305	1100	3900
Rate of cooling p = °C. per min.	9·4	2·3	·55	·13	·03

A factor of safety of 2 has been applied to the above figures.

Table VI C

Values of s_a and n for
L.B.C.
to make total re-annealing time a minimum
$D = \cdot 029$ $d = 9$

Thickness of plate in cm. ...	·5	1	2	4	8
Value of b in cm.	·25	·5	1	2	4
s_a	·007	·007	·007	·007	·007
n ° C. above θ_0	+43	+ 26	+ 9	− 8	− 25
t_a in mins.	10	38	142	520	1950
t_c in mins.	29	106	370	1300	4300
Total time in mins.	39	144	512	1820	6250
Rate of cooling p = °C. per min.	5·0	1·2	·30	·07	·017

A factor of safety of 2 has been applied to the above figures.

Table VI D

Values of s_a and n for
H.C., B.S.C., and L.F.
to make the total re-annealing time a minimum
$D = \cdot 025$ $d = 10$

Thickness of plate in cm. ...	·5	1	2	4	8
Value of b in cm.	·25	·5	1	2	4
s_a	·007	·007	·007	·007	·007
n ° C. above θ_0	+45	+ 27	+ 9	− 9	− 27
t_a in mins.	13	45	150	570	1870
t_c in mins.	24	91	312	1140	3650
Total time in mins.	37	136	462	1710	5520
Rate of cooling p = °C. per min.	5·9	1·4	·35	·08	·02

A factor of safety of 2 has been applied to the above figures.

It now appears from the tables that for practical purposes it is hardly necessary —except perhaps in the case of D.B.C.—to make differences for the various types of glass. Such differences as occur are easily covered by the "factor of safety."

It is therefore suggested that the following scheme would cover practical cases of re-annealing.

Table VII

Thickness of plate in cm. ...	·5	1	2	4	8
Annealing temp. in ° C. above θ_0	40	25	10	− 5	− 20
t_a (annealing time) in mins. ...	10	40	150	550	1900
Cooling rate as far as θ_0 − 100° C. in ° C. per min.	6·0	1·2	·3	·08	·02

If a lower standard of annealing is allowable, as, for instance, in plate glass annealing, the annealing temperature given above will remain unchanged, but the rate of cooling will be increased with consequent reduction in the cooling time.

The author's acknowledgments are due to Mr H. V. Mallison, M.A., B.Sc., for his kindness in reading through and checking the mathematical formulae, and to Sir Herbert Jackson, K.B.E., and Mr H. Moore for their friendly criticisms.

DISCUSSION

Mr F. F. S. Bryson: Mr Twyman obtained from experimental results a law connecting rate of release of strain with temperature. Although later experimental work by other investigators has shown that the law is valid only within certain ranges of temperature, the enunciation of such a law had the effect of directing attention to the necessity for a theoretical study of the various factors involved in the annealing of glass. The theoretical investigations carried out by Mr Hampton considerably extend our knowledge of the relations existing between these factors, and lead to rules, the application of which should be of considerable use in practice. I should have liked, however, to see the results of tests applied to determine whether the degree of annealing obtained by adhering to the scheme detailed by Mr Hampton at the end of the paper corresponds to the standard required in practice.

Mr T. Smith: The importance of a thorough knowledge of the principles to be employed in annealing is too obvious to users of optical glass to need any advocate; in particular the determination of the most economical conditions under which annealing can be carried out, in so far as it may influence the price of such costly material, will be widely appreciated. The numerical results quoted in the paper appear to be based entirely on work in America; the publication of results obtained in this country with British glasses is very desirable, both as confirmation of the general results obtained by other workers, and because definite information concerning the physical properties of particular glasses which are commercially reproducible with unusual precision encourages their employment for other purposes than those for which they are normally produced. It is almost superfluous to call attention to the value measurements of these properties to a high standard of accuracy are likely to have in pure science.

Without necessarily impugning the general character of the conclusions reached in the paper, the mathematical discussion as it stands seems to me unconvincing. It would be much strengthened if it were remoulded, paying special attention to the dimensions of the quantities introduced; numerical agreement with experimental results is insufficient. The main difficulty of the paper, however, is due to the adoption of formula (4), which Adams and Williamson proposed. A little consideration will show that this formula is quite unsuitable for the representation of the facts. Reference to the paper in which it was first proposed shows a complete failure to keep the physical facts in mind in dealing with this aspect of the investigations; it is not therefore surprising to find that deductions based on this formula are decisively disproved by the observations. To the extent to which this law fails, the theory of this paper also will tend to be erroneous. While the experience gained in applying the theory in its present form may be conclusive in showing that the practice of annealing is generally on a satisfactory basis, the subject appears to be of sufficient interest to justify fuller theoretical discussion than is possible here.

Mr Hampton: I should like to take this opportunity of recording my appreciation of the pioneer work on the subject of annealing carried out by Mr Twyman.

Although the law deduced by him is not a complete explanation of the facts, he did valuable work on this subject at a time when nothing was available other than meagre empirical knowledge, and later work has been only an elaboration of his theories.

The reason for utilizing American data is simply that these glasses are the only ones for which complete lists of physical properties are available, though it is difficult to form an opinion as to the reliability of some of the values given, such as, for instance, the thermal conductivity.

The theory advanced in this paper is only put forward as a rough approximation to the truth, as many assumptions, which are obviously only partially correct, have had to be made. It is known that the interval in which the velocity coefficient λ doubles itself is not constant. This is shown by Adams and Williamson* and English†.

Again, the coefficient of linear expansion is not constant, as assumed for the sake of simplicity‡.

With regard to Mr Smith's criticism, I must point out that no physical basis is claimed for equation (4) and that other formulae may be put forward which will also agree with the numerical values found experimentally. I submit, however, that this does not invalidate the utility of the general conclusions and that the formulae are useful in the same way as Planck's law for the distribution of spectral radiation is useful, although it is not in agreement with electro-dynamical principles§.

I regret that I do not understand Mr Smith's statement that the "deductions based on this formula are decisively disproved by the observations." To the best of my knowledge the results are qualitatively, at least, as would be expected from the formula.

It must be generally accepted that the subject is worthy of fuller investigation, both theoretically and experimentally, and that the results of such inquiry must benefit both practical glass manufacture and pure science.

* Adams and Williamson, *Journ. Franklin Inst.* 190 (1920) 597, 835.
† English, *Journ. Soc. Glass Tech.* 8 (1924) 206.
‡ See Peters and Cragoe, *Journ. Opt. Soc. Amer.* 4 (1920) 105.
§ See Griffiths, *Methods of Measuring Temperature* (1918 edition) p. 107.

THE BACK VERTEX POWER OF A COMBINATION OF LENSES

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ABSTRACT. The general relations between the constants of a compound instrument and those of its components are obtained when the imagery is collinear. From these relations the expression for the vertex power of a compound instrument is built up in the form convenient for application in spectacle calculations. The formulae have other applications.

Introduction

In recent years it has come to be generally realized that in the fitting of spectacles the back focal length, that is the distance from the lens surface nearest to the eye to the point at which parallel incident rays are brought to a focus, is of more fundamental importance than the true focal length of the lens; or, in terms of powers, that the back vertex power, and not the true power of the lens, is the significant quantity. It fortunately happens that, in the simple lenses appropriate for correcting defects of vision, the difference between these two powers is small, though not in all cases entirely negligible, so that the substitution of the more logical system can be made with very little alteration in earlier practice. As the theory of lenses in general has been built up on the assumption that the quantities hitherto regarded as fundamental should be adopted as the basis of measurements, some slight recasting of a few well-known formulae is not inappropriate, and a number of papers dealing with some of these have been contributed to the technical papers more particularly concerned with this side of optics. Of these formulae that which gives the back vertex power of a combination of lenses in terms of the properties of the various component lenses is probably of the greatest importance, since the use of two or more lenses in contact or approximately in contact is usual in sight testing. The discussions of this problem which the writer has seen only consider special cases, and the simplicity of the general formula seems to have been overlooked. It may therefore not be without interest to consider this fundamental relation from first principles.

The general formulae for the combination of two lenses

The elementary theory of optical instruments is based on the fact that corresponding to any given object point on the axis of the instrument, there is one, and only one, definite image point, and conversely that corresponding to any given image point there is one, and only one, object point. This unique relationship is expressed mathematically by an equation which only involves the first power of the quantities which determine the positions of the object and image points. Let a point be chosen arbitrarily on the axis from which the position of any object point is to be measured. Denote the distance of the object point from this selected point by x, and let x be considered positive when the incident light reaches the chosen reference point before the object point, and negative when the light reaches the object point before the reference point. Similarly let another point be chosen on the axis from which the positions of image points will be measured. The distance of the image point from this second reference point will be denoted by x', the same convention as to signs being adopted as with x. If the image point determined by the quantity x' is the image of the object point determined by the quantity x, the relation between these two quantities is

$$axx' + bx' - cx - d = 0 \qquad \ldots\ldots(1),$$

where a, b, c, d are some constant quantities, their actual values being likely to depend on the properties of the particular instrument and on the particular positions which have been chosen for the two reference points. Although the equation involves four of these constants, they are equivalent to only three, viz. the ratios of any three of them to the fourth, since the equation will give exactly the same values for x' when a given value is assigned to x if every coefficient is divided by the same quantity. If the whole equation is multiplied by a, it may be rearranged in the form

$$(ax + b)(ax' - c) + bc - ad = 0 \qquad \ldots\ldots(2),$$

from which it is at once evident that a change in the positions chosen for the axial reference points, which involves the addition of constant lengths to x and x', causes changes in at any rate the values of b and c, but that a and $bc - ad$ are not changed. It is also obvious from this form of the relation between x and x' that $bc - ad$ is necessarily a finite quantity, for otherwise the connection between x and x' disappears. These two properties of $bc - ad$ have suggested that the uncertainty in the values of the four constants in equation (1) may be removed by adopting the arbitrary convention that $bc - ad$ shall have a special value which shall be the same for every instrument. Any finite value is theoretically permissible, but it happens that one particular value is much more convenient than any other.

To realize this, consider a compound instrument obtained by causing the light to pass through two systems in succession. Let the first system be that considered above, the properties of which are contained in the values of the four constants a, b, c, d, the distances of any pair of conjugate points from arbitrary reference points being x and x'. Let the corresponding four constants for the second system be α, β, γ, and δ, and the distances of conjugate object and image points from any

pair of reference points on the axis be ξ and ξ' respectively. For the complete instrument the natural reference points will be those from which x and ξ' are measured. The quantities x' and ξ are not of interest for the complete instrument, and they are easily removed from the equations. Let t be the distance between the image reference point for the first system and the object reference point for the second system, being considered positive when the light travelling through the instrument reaches the former before the latter. Then the distances x' and ξ of any point from these two reference points are evidently connected by the equation

$$x' - \xi = t \qquad \ldots.(3),$$

and since (1) and the corresponding equation for the second component

$$a\xi\xi' + \beta\xi' - \gamma\xi - \delta = 0 \qquad \ldots.(4)$$

give for x' and ξ the values

$$x' = \frac{cx + d}{ax + b}, \quad \xi = -\frac{\beta\xi' - \delta}{a\xi' - \gamma},$$

the required relation is by (3)

$$\frac{cx + d}{ax + b} + \frac{\beta\xi' - \delta}{a\xi' - \gamma} = t,$$

or

$$(a\beta + c\alpha - aat)\, x\xi' + (b\beta + da - bat)\, \xi' - (a\delta + c\gamma - ayt)\, x - (b\delta + d\gamma - byt) = 0.$$

This is the relation for the complete lens corresponding to (1) and (4) for the two components. If capital letters are used for the constants of the whole instrument, this equation is equivalent to

$$Ax\xi' + B\xi' - Cx - D = 0 \qquad \ldots.(5)$$

and the corresponding coefficients are proportional to one another.

A convenient convention

It is obviously convenient so to arrange any conventions that the four characteristic coefficients of the combined system of lenses are derivable from those of the components in the simplest possible way. For example, fractional expressions should be avoided, also unnecessary numerical factors. These conditions are met by requiring the last two equations to be identical with one another, so that the new coefficients are given by the equations

$$\left.\begin{aligned} A &= a\beta + c\alpha - aat \\ B &= b\beta + da - bat \\ C &= a\delta + c\gamma - ayt \\ D &= b\delta + d\gamma - byt \end{aligned}\right\} \qquad \ldots.(6).$$

These values of A, B, C, D may be used to find the value of $BC - AD$ in terms of the coefficients of the components. The result is easily found to be

$$BC - AD = (bc - ad)\,(\beta\gamma - a\delta).$$

As each of these factors is to be equal in value, the only values consistent with the expression of the addition laws in the simplest possible form (6) are 0 or + 1.

The former must be rejected as a finite value is essential, and the necessary convention is thus

$$bc - ad = 1 \qquad \ldots.(7).$$

This convention would be meaningless if $bc - ad$ were not of zero dimensions; that is to say it must be a mere number, and not for instance a length or an area. Now from equation (1) it is seen that b and c are of the same dimensions, since they each multiply a length, and every term of an equation that has any meaning must be of the same dimensions. It at once follows that both b and c are pure numbers, that d is a length, and that a is a curvature. In terms of familiar quantities a is the true power of the lens, b and c are magnifications for the two reference points, and d is their reduced separation, but with its sign changed.

The principal point formulae

The foregoing equations will not perhaps be recognized by everyone as the familiar equations of our textbooks, and yet they are these common equations in their most general form. The reason for the difference in appearance is that in our books a very special system of reference is adopted. In the first place the two reference points from which x and x' are measured are taken to be a pair of conjugate points, so that their reduced separation is zero. This means that d disappears from the equations. From (7) this requires the product of b and c to be equal to unity. To secure the utmost symmetry the values of b and c are made equal to one another, the choice being $b = c = +1$. On dividing (1) throughout by xx' and transposing, it takes the familiar form

$$\frac{1}{x'} - \frac{1}{x} = a,$$

and on substituting 1 for b, c, β, γ, and 0 for d and δ in (6) these equations become

$$A = a + \alpha - a\alpha t,$$
$$B = 1 - at,$$
$$C = 1 - at,$$
$$D = -t,$$

of which the first at least is familiar to everyone as the usual expression for the power of the combination of two lenses where t is the separation of their principal points.

The general formula for the wave curvature at the back reference point

Instead of referring measurements to the principal points, the vertices of the surfaces, and particularly the vertex of the last surface, are to be employed for spectacle purposes. This requires the use of equations other than the specialized forms given in the textbooks. As those given above include all possible cases, they may be utilized without hesitation for the investigation of the formulae required in spectacle calculations. Whatever the reference points may be, equation (1) shows that when the object is at an indefinitely great distance from the reference

point, so that $x = \infty$ the distance x' of the image point from the second reference point is equal to c/a; so far as the position of the principal focus is concerned the system is therefore equivalent to a thin lens of power a/c located at the reference point for the image space. The formula required thus gives the value of A/C in terms of the constants of the component lenses. Now equations (6) at once give

$$\begin{aligned}\frac{A}{C} &= \frac{a\beta + c\alpha - a\alpha t}{a\delta + c\gamma - a\gamma t} \\ &= \frac{\alpha}{\gamma} + \frac{a(\beta\gamma - \alpha\delta)}{\gamma(a\delta + c\gamma - a\gamma t)} \\ &= \frac{\alpha}{\gamma} + \frac{a}{c} \times \frac{c}{\gamma C} \qquad \ldots\ldots(8),\end{aligned}$$

$\beta\gamma - \alpha\delta$ disappearing from the numerator in consequence of (7). The quantities a/c, α/γ, and A/C are the vertex powers of the first and second components and of the complete system respectively if the reference points for the image spaces are placed at the vertices of the last surfaces of the components, and $1/c$, $1/\gamma$, and $1/C$ are the so-called "effectivity" factors—not a very suitable name. Equation (8) then states that the vertex power of the complete system is to be found by adding the vertex power of the second part of the system to the product of the vertex power of the leading part of the system with a quantity which involves only the effectivity factors of the complete system and of its parts. The inclusion of the effectivity factor of the whole system is open to objection, but the formula in this shape is useful inasmuch as in many cases this value will be known with ample accuracy without any calculations. When this is not the case the formula is modified by finding an expression for the ratio of the effectivity factors of the leading component and of the whole lens. From the third of equations (6),

$$\frac{C}{c} = \gamma - \frac{a}{c}(\gamma t - \delta) \qquad \ldots\ldots(9),$$

showing that nothing need be known about the leading component but its vertex power. The front vertex powers are entirely absent from the equations, and the leading reference points can therefore be placed wherever it is convenient to place them in order to interpret the meaning of the equations without further modification. Let this point for the second part of the system then be placed at the image of the back vertex or rather at the image of the reference point in the final image space wherever that may be. Then δ becomes zero, and t is the reduced separation between the image reference points for the two parts of the system. Denote this special value of t by T, reserving the small letter for the more familiar meaning of the distance between two successive surfaces of the system. If the notation be now modified by writing V_1, V_2, V_{12} for the vertex powers of the first component, the second component, and the complete lens respectively, and e_1, e_2, e_{12} for their effectivity factors, the general formula takes the simple form,

$$V_{12} = \frac{V_1 e_2^2}{1 - V_1 T} + V_2 \qquad \ldots\ldots(10).$$

Formula for any number of component lenses

This can at once be extended to the case in which there are any number of components. Let suffixes be added to the T's to show between which reference points they are measured—so that for example in the above formula T stands for T_{12} because it represents the reduced distance between the reference points for lenses 1 and 2. The formula for deriving the vertex power of the system obtained by adding a third lens to the system considered above is

$$V_{13} = \frac{V_{12}e_3^2}{1 - V_{12}T_{23}} + V_3,$$

and the general procedure for extending this to any number of lenses is obvious. In applying these expressions it will usually happen that each component is a single lens. In each case the vertex power and the effectivity factor should be known. In addition, information is required about the separations of the reference points. In the case of spectacles these are the vertices of the surfaces, a choice which seems to be dictated by the accident that clearance for the eyelashes is the factor which limits the nearness of the lens to the eye. This separation is in part due to the distance between the successive lenses, and must be obtained by the oculist. The remaining part is a constant of the following lens of the pair, and may be given in a tabular statement of the properties in the case of trial case lenses. It is equal to $e\tau$, where τ is the axial thickness of the component divided by the refractive index of the glass. From the general formula it is evident that the most useful particulars to have tabulated will be V, which is the denomination of the lens; e^2, the correcting factor to apply to the vertex power of the preceding lenses when two or more are used as a single system; and $e\tau$, the part of the reduced separation due to the thickness of the lens. As the air separation of lenses in the trial frame is usually so small as to be negligible for all normal purposes, the last quantity will normally be the complete reduced separation.

The result given in equation (10) is of course applicable much more widely than the formal title of this note indicates. For instance, in the construction of optical instruments in general the positions of lenses are determined by their rims, and not by the positions of their external vertices. The planes of their rims can be adopted as reference planes from which all axial measurements are made, and calculations can be made on the basis of these reference points instead of the more usual principal points.

Convenient approximate formulae

In many cases an approximate value for the vertex power is sufficient. This is invariably the case for spectacle purposes. A suitable expression for the vertex power under these circumstances is

$$V_{1n} = \Sigma V_h e^2_{h+1} e^2_{h+2} \dots e_n^2 + \Sigma T_{h,h+1} (V_1 + V_2 + \dots + V_h)^2 \quad \dots . (11),$$

where the summations are to extend from $h = 1$ to $h = n$. The well-known expressions for the vertex power of a single lens are obtained from (10) and (11) on putting $n = 2$ and $e = 1$.

The thickness of the lenses has up to the present been treated as an independent variable; in this respect the general practice of writers on spectacles has been followed. It is difficult to find an adequate reason for this practice as the thickness is only introduced to enable the desired aperture to be attained, the small addition necessary for mechanical strength being negligible in its optical effects. A rough approximation to the thickness of the more powerful positive lenses is adequate for giving the thickness corrections required. The simplest of all formulae for the thickness may be adopted to remove the explicit presence of the thickness from all spectacle formulae. This rule states that in positive lenses the semi-aperture is a mean proportional between the thickness and the focal length. This may be translated for vertex formulae into the approximate expression $\tau = kV$, where k is a constant which will be about ·0003 or ·0004 for spectacle lenses of the ordinary diameter when the metre is taken as the unit of length and the dioptre for the unit of power. For negative lenses k is to be taken as 0. This value would enable the second sum in equation (11) to be written as

$$k\Sigma V_{h+1}(V_1 + V_2 + \ldots + V_h)^2$$

in the case in which all the lenses are positive and are in contact with one another. As another instance of the convenience of formulae in which this substitution has been made, that for the power of the leading surface of a lens of vertex power V with a base curve of power B may be taken. The power required is F, where

$$F = \frac{V - B}{1 + kB(V - B)}.$$

THE THEORY OF NEUTRALIZATION

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ABSTRACT. Reasons are given for rejecting the current theory of neutralization, and a new theory which represents the experimental procedure more exactly is propounded. It is shown that the test should be made with a large separation between the test object and the observer's eye, and that the connection between the vertex powers of the lenses when neutralization is exact involves the distance of the test object and the order in which the lenses are placed. From the discussion of numerical results it is concluded that the simple law of the zero sum for the vertex powers is attained with ample accuracy as a mean result, the combination of high powered lenses of equal and opposite vertex powers appearing as a positive lens from one side and negative from the other. The theory leads to the suggestion that in practice plano-lenses should be tested with their plane, not their curved, surfaces in contact.

The practice and present theory of neutralization

Neutralization of one lens by another of known power and opposite sign is a widespread method of finding the power of an unknown lens, and is particularly useful in the case of spectacles and eyeglasses. The simplicity and rapidity of the process have rightly commended it for this province of optics as a standard method, and the extensive experience that is available of the accuracy and reliability of the measurements that it yields can leave no doubt as to its suitability for the purpose. Naturally the method is described, and its theory discussed, in the many books dealing with this branch of optics. An examination of these reveals considerable differences in the recommendations made as to the practical application of the method. For instance, in some cases it is recommended that the lenses should be held close to the eyes, in others that they should be at some distance away so that the edge of the lens which serves as a dividing line between the object examined and its image can be clearly defined. Again in some instances it is recommended that the object viewed should be at a considerable distance from the observer, and in others that an object only two or three feet away should be chosen. Despite these obvious differences there is agreement on two important matters. All writers give as the practical criterion that neutralization has been properly secured the continuity of the object and its image as seen at the edge of the lens as this is displaced in a plane normal to the line of vision; and they all give as the explanation that this condition of affairs is brought about by the parallelism of the rays before and after refraction, usually adding a diagram showing rays parallel to the axis.

On this explanation it is considered obvious that the distance of the object examined is in essence immaterial, and so will be chosen as proves most convenient. The diversity in advice as regards the practical application of the method has thus excited no surprise. It is, however, not a little curious that this explanation should be coupled so generally with an everyday method of test, when in fact the theory has nothing whatever to do with the experimental method employed, but relates to a method of testing which makes use of entirely different principles.

Objection to the theory

That the theory does not fit the method is evident if the use of a near object is considered. It is known that the lenses of ordinary form which neutralize differ in power, the difference amounting at 20 D to about 10 per cent. The telescopic system which the two lenses are supposed to form when neutralization is secured is thus not of unit power, and therefore the object and its image do not lie on the same straight line parallel to the axis. That is to say, the theory is inconsistent with the attainment of neutralization except in the special case when the two true powers are equal and opposite and also the two vertex powers are likewise equal and opposite. It is necessary, therefore, to construct an acceptable theory of neutralization, but at the outset it is known that the accurate theory will yield results which differ very little in value from those given by the theory which must now be rejected. To this extent the difference may perhaps be regarded as of theoretical interest only.

The new theory

From the statement of the criterion employed to determine when neutralization has been attained, it is evident that the line joining the object and image remains parallel to the axis of the lenses as these are displaced. That is to say, the object and image are equal to one another; this is only attained in the strict mathematical sense if the object and image lie in the unit planes of the combined system. Since objects at a finite distance must be employed, and since the unit planes of a telescope are at an infinitely great distance away, it is evident that the neutralizing systems together form a telescope only in the exceptional case mentioned above. Strictly speaking the exact position of the object employed in the test is of moment, but practically the method is rendered compatible with reasonable freedom by the fact that in the neighbourhood of the unit planes the magnification differs little from unity. The extent of "neighbourhood" in this connection is proportional to the focal length of the combination, for which a large value is obtained. The compatibility of the experimentally observed facts with the theory is thus seen in general terms. To enable any precise comparison to be given, the new theoretical equations must first be obtained.

Relation between neutralizing powers

The notation employed in a previous paper* may be adopted for this investigation, and the general relations there given between the constants of a compound system and those of its parts will enable the indications of the neutralizing test to be expressed in terms of the quantities of practical importance. Merely as a matter of convenience to avoid the presence of a number of negative signs, the eye will be assumed to lie in what has been taken as the image space in that note, so that the object employed in the test is in the image space. If exception is taken to this arrangement, it is only necessary to substitute $-x'$ everywhere for x and $-x$ for x', and to suppose the two lenses turned so that their order is changed. Taking A, B, C, D as the four Gaussian constants of the combination, the distance of the image plane, where the magnification is m, from the image reference point is x', where

$$x' = \frac{C - m}{A}.$$

In expressing this in terms of the constants of the component lenses the equation for A

$$A = a\beta + c\alpha - ac\alpha t$$

suggests that the quantities to be used should be the back vertex power of the first lens (reckoning of course in this case from the eye) and the front vertex power of the second lens. Denote these by V_1 and U_2 respectively, so that

$$V_1 = \frac{a}{c}, \quad U_2 = \frac{\alpha}{\beta}.$$

The equation for A now takes the form

$$Ae_1e_2' = V_1 + U_2 - V_1U_2t,$$

where e_1 and e_2' are the effectivity factors corresponding to the two vertex powers V_1 and U_2 respectively. The formula for C is

$$C = a\delta + c\gamma - a\gamma t$$

or

$$Ce_1 = \gamma(1 - V_1t) + \delta V_1.$$

Now the origin from which the external distance x' is measured may be selected wherever is most convenient. Let it be the image of the front vertex of the second lens. Then the two reference points for this lens are object and image of one another, so that $\delta = 0$. The relation $\beta\gamma - \alpha\delta = 1$ then gives $\gamma = \beta^{-1} = e_2'$, and the equation for x' is

$$\frac{x'}{e_2'} = \frac{e_2'(1 - V_1t) - me_1}{V_1 + U_2 - V_1U_2t}.$$

It appears from this equation that, if the two lenses were in contact at their internal vertices and the value of x' were great, the denominator on the right would vanish and the ordinary rule would apply. In fact this argument assumes no loss of

* "The Back Vertex Power of a Combination of Lenses." *Trans. Opt. Soc.* 26 (1924–25) 31.

sensitivity with large values of x', and its validity has yet to be established. The formula has thus to be used in one of the forms obtained by replacing m by unity, of which the more convenient is obtained by regarding the lens remote from the eye as the unknown, and is

$$U_2 = -\frac{V_1 + e_1 e_2' X'}{1 - V_1 t} + e_2'^2 X' \qquad \ldots\ldots(1),$$

where X' is written for the reciprocal of x'. Before considering this equation it will be desirable to obtain an expression for the sensitivity of the test, which obviously depends on the angle between the object and its image as seen at the eye.

Sensitivity

Suppose that the lens is displaced sideways through the distance y'. With reference to the lens this is a displacement of the image point from the axis, and the corresponding displacement of the object point is y'/m. The displacement of the line seen by the eye through the lens from the line joining the eye to that seen by direct vision is thus $y'(1 - 1/m)$. Let the distance from the eye to the object space reference point for the first lens be z, so that the distance from the eye to the line seen through the lens system is $z - \dfrac{Bx' - D}{Ax' - C}$. If the angular displacement is denoted by θ, the product of the last found distance and θ gives another value for the linear departure from collinearity, and the equality of the two expressions found for this displacement gives the relation

$$\frac{\theta}{y'} = \frac{Ax' - C + 1}{Azx' - Bx' - Cz + D} \qquad \ldots\ldots(2).$$

The fraction on the left of this equation is a measure of the sensitivity of the method of neutralization in the most general case. For the purposes of this discussion it will be noted that y' cannot exceed a value which is determined by the half diameter of the lenses, and will in practice be appreciably less than this maximum value. A convenient value to assume is 1 cm. On the other hand departures from exact neutralization will escape detection unless θ exceeds a certain value. Now it is known that under good conditions the eye can detect want of collinearity of two straight lines of exceedingly small amount, much smaller than the distance of resolution. Such values are not admissible in this problem, because, in addition to the failure of other conditions, the two lines seen are at different distances from the eye. It seems appropriate to assume that the minimum value for θ will be somewhat greater than the angular resolving power of the eye. The value adopted in the illustrations which follow is $\pm$ ·001 radian, so that the left side of equation (2), which may be denoted by the single symbol ϵ, has for extreme values $\pm$ ·1 D.

No assumption has yet been made as to the position of the object space reference point for the leading lens. It is convenient to retain symmetry in the treatment of the two lenses by selecting the point conjugate to the internal reference point,

so that $d = 0$. The equations for the constants of the combination may then be written

$$Ae_1e_2' = V_1 + U_2 - V_1U_2t,$$

$$B\frac{e_2'}{e_1} = 1 - U_2t,$$

$$C\frac{e_1}{e_2'} = 1 - V_1t,$$

$$D\frac{1}{e_1e_2'} = -t.$$

In substituting these expressions in equation (2), write $U_2 + u$ for U_2, and give U_2 the value assigned by equation (1). Then u is the error liable to be made in the determination of U_2 owing to the inability of the eye to detect very small relative movements of the two lines. The relation between ϵ and u obtained in this way is

$$\epsilon = \frac{Ax' - C + 1 + Cx'v}{Azx' - Bx' - Cz + D + (Cz - D)x'v} \qquad \ldots\ldots(3),$$

where

$$v = \frac{u}{e_2'^2}.$$

Inserting now the value of U_2 given by (1), which is equivalent to

$$Ax' - C + 1 = 0,$$

the expression for the possible error in the power becomes

$$v = \frac{\epsilon X'(z + Bx' - D)}{C + \epsilon(Cz - D)} \qquad \ldots\ldots(4).$$

The best distance of the test object

In dealing with this equation it will usually be permissible to neglect quantities which may have to be retained in equation (1). It will be assumed here that the differences between the effectivities and unity may be neglected and that the separation between the two lenses is to be replaced by zero. This gives the approximate value

$$u = \epsilon(1 + X'z).$$

The important factor in this expression is the ratio of the distances from the eye and from the lens to the test object. Since both x' and z are positive in the application under consideration, the possible error is never less than ϵ, and this value is attained when the object is many times farther from the lens than is the eye. This conclusion seems to be in satisfactory agreement with experience. When the lens is half-way between the eye and the object, the uncertainty is 2ϵ. Since z must not be reduced to such an extent that the test line, its image, and the edge of the lens cannot be seen clearly at the same time, the use of a distant object is indicated.

It will often happen that circumstances enforce the use of a near test object. Instead of the negligible value of X' which is secured by the use of a distant object, a value of two or three dioptres has then to be faced. In the numerical examples

which follow the value 2D is adopted. This large value may prevent the universal employment of the simpler special correcting term

$$-X'V_1t$$

in place of the terms in (1) involving X'. There appear to be only two alternatives open—either to employ a more complex formula than heretofore, or to admit the possibility of considerable error in the value deduced for the power of the unknown lens. The difficulty can perhaps most readily be avoided by tabulating the corrections in the cases of frequent occurrence where the errors are apt to be large.

The avoidance of distortion

Calculation of neutralizing powers by equation (1) shows that in practical cases the greatest departure from the simple addition law arises from want of contact of the two lenses on the axis. For this reason it is important to secure such contact in the application of the method. A more important reason is that in the absence of such contact, or more generally in the presence of strongly opposed curvatures as when the curved surface of a plano-concave lens is placed against the flat side of a plano-convex, violent distortion is present, and this offends against a condition tacitly assumed in the very basis of the method. The absence of such strongly opposed curvatures implies a small curvature difference at the contact surfaces, so that even where axial contact is not attained the correction due to the separation is small. Under these conditions the corrections due to the thickness and to the finite value of X' may be considered independently of one another.

Direct and reversed neutralization

If the suffixes of the earlier equations are abandoned, and likewise the accent to X, equation (1) states that, if V is the vertex power of a known lens and U that of an unknown lens, the vertices to which these powers relate facing one another when neutralization is effected, and if the effectivity factors of the known and the unknown lenses are respectively e and e',

$$U = -\frac{V + ee'X}{1 - Vt} + e'^2X$$

when the known lens is next to the eye, and

$$U = -\frac{V + ee'X - e^2X}{1 - Vt + e^2Xt}$$

when the unknown lens is next to the eye. Denote the value of U given in the latter case by U'. Then, when $t = 0$,

$$U + U' = -2V + (e - e')^2 X$$

and

$$U - U' = -(e^2 - e'^2) X.$$

The first equation shows that the finite value for the distance of the object examined is to produce a positive value for the sum of the vertex powers of the neutralizing lenses. This excess varies approximately as the fourth power of the strength of either lens, and is about four times as great with plano-lenses placed

with their flat sides in contact as with equi-curved lenses of the same powers and apertures. Its value in the case of 20 D plano-lenses, however, only amounts to about ·02 D, and may thus be safely neglected in all cases. The second equation shows a difference between the neutralizing powers obtained with the two possible orders in which the lenses can be arranged with respect to the line of sight. This difference varies approximately as the square of the power of one of the lenses, and is twice as great with plano-lenses placed with their curved surfaces outwards as with equi-curved lenses of corresponding dimensions. The direction of the difference may be expressed by saying that, when neutralization is exactly secured, the sum of the vertex powers of the two lenses tends to be positive if the combined lenses resemble a meniscus lens with its concavity towards the eye; on the contrary the sum tends to be negative if the convexity is turned towards the eye. Unlike the previous term the value of this difference is not always negligible, and its approximate value in special cases may be of interest. With the Optical Society's Standard equi-curved lenses of power 20 D the correction to the nominal zero sum for the vertex powers is ± ·13 D, and with the smaller diameter plano-lenses suggested by Mr Swaine in a recent paper* read before the Society the value is ± ·25 D. With equi-lenses of larger diameter this value would be increased nearly in proportion to the square of the diameter.

Neutralization of plano-lenses

In the paper just mentioned Mr Swaine assumes that neutralization with plano-lenses will be carried out by placing the curved faces together, an assumption generally made when plano-lenses are considered. This course offers the advantage of dispensing with a correcting term, but suffers from the much more serious defect that the powers neutralized in this way are not those required in practice, so that the neutralizing power of the positive lenses has to be tabulated as a quantity distinct from the useful vertex power. From the foregoing discussion it is clear that this course is unnecessary. The corrections instanced above when the external surfaces are curved are extreme values, and rapidly fall as the power is reduced. Their tabulation is thus a very simple matter. Indeed even this tabulation may be dispensed with, it being merely necessary with the highest powers to see that the observed departures from strict neutralization are of one sign when the positive lens is nearest to the eye, and of the opposite sign when the lenses are reversed without being separated.

Neutralization of deep meniscus lenses

For verifying deep meniscus lenses the method suggested by Mr Swaine of obtaining a similar lens in the trial frame by the combination of a plano-convex and a plano-concave lens, and seeing that the actual spectacle lens is neutralized by the same lens as this combination, is unexceptionable. The alternatives are either to use special neutralizing lenses with the complementary base curve, when the foregoing theory applies as it stands, or to neutralize the front vertex power of

* *Trans. Opt. Soc.* 25 (1923–24) 73.

the meniscus lens and calculate the back vertex power. For negative meniscus lenses no new problem is introduced, as the difference between the back and front vertex powers is negligible. When the meniscus is positive the convex surface will be placed in contact with the concave surface of the negative used for neutralizing, and the preceding analysis then shows how the front vertex power of the meniscus lens is obtained by the neutralizing test. It will be observed that $e = 1$ and with sufficient accuracy $e' = 1 + B\tau$, where B is the power of the base curve and τ is the reduced thickness of the meniscus lens. The back vertex power W of this lens is perhaps obtained most conveniently from the formula

$$W = \frac{U}{e'^2 - e'\tau U}.$$

An approximate formula derived from this exact expression is

$$W = \frac{U}{1 - \tau(U - 2B)}$$

and this again may be simplified by the substitution for τ of the approximate value kU, as in the note to which reference has already been made. The correcting term in the denominator is obviously by no means negligible, for U is positive and B is negative, and usually either -6 D or -9 D. If figures are substituted representing the lenses actually made, it will be seen that the conclusions reached in the previous cases as to the accuracy of the results given by the method apply equally to meniscus lenses.

It should be noted that in the above equation e' is not the back vertex effectivity factor. This factor could have been employed, but the foregoing method of calculation seems preferable for this special purpose. It may also be noted that the front vertex power of the trial lenses is not required for any neutralizing tests.

The actual formula should give the back vertex power of the meniscus lens in terms of the back vertex power of the neutralizing lens. Since as a mean result it has been seen that $U + V = 0$, the formula becomes

$$W = \frac{-V}{1 + \tau(V + 2B)},$$

or

$$W = -\frac{V}{1 - kV(V + 2B)},$$

where k may for practical purposes be regarded as a constant, the value being about ·0004 when V is expressed in dioptres. If the unit of length is changed, it is only necessary to bear in mind that k is an area—actually a multiple of the area of the surface of the spectacle lens—to see how its value will be changed. If a correcting term in the numerator is preferred to a correcting denominator factor, the formula will be

$$W = -V - kV^2(V + 2B).$$

DISCUSSION

Mr. P. F. Everitt: In practice it is possible for a skilled observer to detect a departure from neutralization of ·05 D or even less.

Mr Smith: It should be observed that two assumptions were made in deriving an approximate value for ϵ. The values Mr Everitt mentions are of the same order of magnitude as that indicated in the paper. For the purposes of this discussion the precise values are not material.

REVIEWS

Factory Costing.—A Textbook for Students and a Reference Book for those concerned with Workshop Production. By H. H. EMSLEY, B.Sc. (Eng.). Pp. vi + 253; 42 Figs. (London: Constable and Co., Ltd., 1924.) Price 7*s.* net.

The author states in his preface that this book "appears in reply to requests to put into a handy form the substance of a course of lectures on the subject of 'Costing' given to students training to become Engineers, Scientific Instrument Makers, and Ophthalmic Opticians."

The scope of the book is accordingly wider than would be the case were it primarily intended as a treatise purely on Costing, and it includes, *inter alia*, some useful notes on the ordering of materials and framing of specifications, and a chapter on Factory Organization and Routine. The book should help both the "practical" man and the cost accountant to appreciate the peculiar requirements and difficulties each of the other and could be read with benefit by many of the younger generation who are inclined to regard their own particular sphere of activity as the central pivot round which the works revolve.

The system of Costing outlined, which is designed primarily with a view to fulfilling the peculiar requirements of instrument and light engineering works, has the virtue of simplicity, and the average student, by a careful perusal of the book, should obtain a sound general idea of the functions of a costing system and the mechanism employed to attain the desired ends. On a first study of the subject of Costing from the average textbook, it is only too easy for the hapless student to be led into a position where he cannot see the wood for the trees.

At the same time, whilst the system outlined could be introduced with little or no modification in small works, or in medium-sized works producing commodities of a homogeneous nature, the mechanism would require further elaboration before it could fulfil all the varied requirements of large works, particularly when the commodities produced are of a heterogeneous character.

In the chapter on Estimating, more stress might have been laid on the desirability of varying the overhead expenses to be charged in an estimate according to the nature and magnitude of the enquiry. Taking an extreme case, an enquiry may be received which shows a prospect of doubling the normal output of the works over a period of twelve months; in such a case it is frequently not only desirable but sound in principle to credit the estimate with the whole benefit of the reductions in overheads which would result from the order; otherwise the order may be lost or, even if it is obtained, the costs for the normal output of the works, as shown by the Costing system, and owing to overheads being averaged over the whole output, will be fictitious and misleading.

The method of dealing with tool costs, though suitable in works where tools are of a general nature, would require modification in works producing commodities requiring specially designed tools for each product. The cost of new tools in such cases often forms a substantial proportion of the total value of an order and it is better practice to charge the cost of tools just as directly to the product as are material and labour.

A useful chapter in the book is devoted to the "Use of Costs Data in Management." One point upon which more stress might be laid is the necessity of compiling accurate waste records. When commodities are produced on automatic or semi-automatic machines, the extra cost of labour, due to correction of defective work, need not necessarily bear

any relation to the loss sustained through excessive waste, especially when the waste so made is liable to be beyond rectification. Straightforward percentages showing the proportion of (*a*) rectifiable and (*b*) unrectifiable work to good work completed are preferable to the indirect method advocated.

Another point which requires further emphasis is that records, if they are to be of real value in the future, should be expressed as far as possible in "absolute" terms, in such a way as to eliminate accidental variations due to fluctuations in rates of wages, etc.

A feature of the book is the addition of a number of well chosen "examples," at the end of each chapter, to be worked out by the student. The book also contains a lengthy bibliography as a guide to further reading, which could be improved by pruning, and a number of well selected tables which should enable the student to realize the advantages of having all statistical information conveniently tabulated and ready for immediate reference whenever required.

W. E. S.

The Structure of Crystals (American Chemical Society Monograph Series). By RALPH W. G. WYCKOFF. Pp. 462; 213 Figs. (New York: The Chemical Catalog Co., Inc., 1924.) Price $6.00 net.

One of the main purposes of this series of monographs published by the American Chemical Society is "to promote research...by furnishing a well-digested survey of the progress already made...and by pointing out directions in which investigation needs to be extended." Even while disagreeing with some of his opinions and many of his conclusions, one cannot but admire the thoroughness with which Dr Wyckoff has set himself to the accomplishment of these two important tasks on behalf of the growing science of crystal structure. The book is divided into two parts, the first of which contains an account of the various tools that are available to the experimenter and, more incompletely, of the different ways of using them, the second consisting almost entirely of the results of crystal analysis classified and, in many cases, recalculated. It is admittedly not intended to be an impartial handbook of methods and results but gives rather "the reaction of a single point of view towards the material of crystal analysis as it now exists." This entirely unscientific attitude should be clearly kept in mind by those who study the book with a view to future work in the subject; a literal acceptance of the author's standard of values would certainly lead the uninitiated astray. The chapter dealing with the Laue photographic method is ably and authoritatively written and is worth careful study by all who are using or who are interested in that particular method. The Bragg ionization spectrometer, however, has been dealt with both summarily and inadequately. All who have learned to use this instrument with care and precision will heartily disagree with the unflattering conclusions drawn concerning it. In spite of his arguments many will remain stubbornly convinced that it is still the best means of obtaining precision measurements, on single crystals, of spacings and of angles between planes, especially when no previous crystallographic information is available. Using modern methods, the "extraneous reflections" mentioned in the text are no less easily recognized and eliminated than are the "hazy" diffraction spots recorded on some Laue photographs. As a result of the author's bias against the ionization spectrometer technique he unjustly rejects from serious consideration (in Part II) many studies of organic crystals, including those of naphthalene, anthracene, resorcinol, and tartaric and racemic acids, which are of undoubted chemical and crystallographic interest. His justification of this rejection on the ground that the published data are insufficient to define uniquely the unit cell adopted is untenable in view of the fact that he has reported results for osmium, nickel oxide, etc., for which *no* data have yet been published. Moreover he cannot lightly ignore the fact that from spectrometer measurements alone correct structures were originally deduced for

sodium chloride, diamond, calcite, and many other crystals, and even the values obtained (from intensity measurements) for the variable parameters have scarcely been improved upon. The careful work of Owen and Preston (1923) on Al—Cu and other alloys is entirely omitted.

The logical treatment of the subject from the point of view of the space-group theory is admirable; we wish, however, that it had not been considered necessary to add yet another nomenclature to those already existing. More space could profitably have been devoted to the various types of Bravais lattice and to their distinguishing spectra, and some of the diagrams in Chapter I, especially 24*a*, 36*a*, 43*a*, and 46, could be considerably improved upon, though on the whole the 213 diagrams and reproductions of photographs are an excellent addition to the monograph, which is well produced and contains but few misprints. The last two chapters give much valuable information on "Non-crystalline Diffraction Phenomena" and on "Applications of Diffraction Information" and will be of particular interest to the general reader. An admirable and complete bibliography of papers published up to 1st January, 1924, is included, together with some useful tables and a subject index.

Those actually engaged in research along these lines will find the monograph both stimulating and useful, and, in spite of the strongly individualistic attitude it reflects, it is certainly a valuable addition to the literature of the subject.

K. Y.

Penrose's Annual: The Process Year Book and Review of the Graphic Arts, Vol. 27. Edited by William Gamble, F.R.P.S., F.O.S. Pp. xv + 142. (London: Percy Lund, Humphries & Co., Ltd., 1925.) Price 8*s*. net.

The present volume contains the following articles: "A Review of Process Work," "The Type of the Aldine Poliphilus," "The Progress of Advertising Art," "The Daniel Press, 1845—1919," "Interference Bands: Their Uses and Graphic Records," "1906—1910—1924, Three Important Dates for Heliochromie," "A Revision of the Three-Colour Process," "Speeding-up Rotary Gravure," "Multi-Colour Intaglio Machines," "Modern Progress in Colour-Sensitizing," "Telephoning a Portrait in Colours," "A New Kind of Photography," "Mechanical Engraving," "A Scaling and Uniform Exposure System for a Process Camera," "A Cold-Enamel Process for Zinc," "The Revival of Collodion Emulsion," "Carbon Positives in the Rotary Photogravure Process," "Printing Ink Bases," "Metallic Printing Inks," "Double-Tone Inks and their Possibilities," "Offset and other Printing Processes," "Internal Relief for External Costs," "The Merry War for Supremacy in Illustrative Fields," "The Photo-Engraving and Illustrating Process: What it means to Advertisers," "Fresh Fields and Pastures New for Offset Lithography," "A Monitory Monograph on Dry Flong Moulding," and "The Wood-Engraved Illustrations of the 'Sixties." The book contains numerous illustrations representing process work during the year 1924, in addition to plates illustrating the articles.

ABSTRACTS OF PATENT SPECIFICATIONS

212198. Day, S. M. SIGNAL LAMPS.

A bulb having two filaments in parallel is adjustably mounted behind two dioptric lenses, the stepped surfaces facing each other. The inner lens is coloured. The lower portion of the outer lens is designed to throw part of the light downwards.

212559*. Optische Anstalt C. P. Goerz Akt-Ges. LENSES.

An optical attachment for diffusing the image produced by an objective is described. The angle of the image is slightly reduced, in order to remove the focussing difference for objects at a finite distance.

212570*. L'Infra Rouge (Procédés Charbonneau) Soc. Anon.
DETECTING INFRA-RED RADIATION.

Infra-red radiation is detected by concentrating it upon a travelling band impregnated with a substance obtained by the reaction of mercuric chloride with silver iodide, when a discoloration lasting for a few seconds is observed.

212999. Hilger Ltd., A., and another. LENSES.

Objectives for ultra-violet work have their components made of rock-salt and crystalline quartz. One form described consists of two separated collective quartz components, with a quartz-rock-salt doublet between them.

213286. Optische Anstalt C. P. Goerz Akt-Ges. REFRACTOMETERS.

Fluid-testing refractometers are fitted with a temperature-compensating device such as a pair of movable prisms, which have to be set according to a given scale.

213373. Parsons, Sir C. A. REFLECTORS.

A search-light reflector is provided with means whereby it may be distorted so as to modify the spread of the reflected light, for instance, so as to produce a fan-shaped instead of a parallel beam.

213674. Hilger Ltd., A., and others. TESTING LENSES.

In using the lens-testing interferometer, a negative lens corrected for spherical aberration is interposed in one part of the divided beam of light, the arrangement being such that parallel incident light emerges as if coming from the point for which the lens is to be corrected. Provision may be made for comparing a standard lens with the one under test.

213947. Bergh, S. J. van den. RANGE-FINDERS.

A telemeter is employed in connection with auxiliary apparatus for predicting the position of an aircraft after an interval which is sufficient to allow for the time occupied in observation, gun-laying and firing, and in the flight of the projectile.

213994. Hasselkus, J. W., and others. PROJECTION ARC-LAMPS.

The electrodes are loaded with a material containing cerium oxide, to allow large currents to be carried without an excessive production of heat.

* Specification published before acceptance.

214002. Coulson, W. T. VEHICLE LAMPS.

The bulb is placed at one of the focal points of an ellipse, and at the other there is supported a semi-opaque screen, adjustable vertically and axially. This point is also the focus of an eccentrically cut projection lens, which forms the front of the lamp.

214227*. Poulsen, A., and another. RECORDING VIBRATIONS.

An optical system is described for producing a photographic record of the movement of a mirror attached to the armature of a magnet in circuit with a microphone. A slit is placed between a source of light—preferably linear—and a projection lens, which throws the light upon the mirror. A cylindrical lens is interposed between the mirror and the photographic film, so as to produce considerable variation in the length of the image which falls across the film.

214485. Chapman, W., and another. MICROSCOPE CONDENSERS.

One of the surfaces of the lens is shaped to the form of a prolate spheroid. An example consisting of two components is described, designed to work at N.A. 1·2, in which the outermost surface is spheroidal.

214638*. Franklin Optical Co. BI-FOCAL LENSES.

The components are heated electrically and fused together in a vacuum, so that no air is trapped between the contacting surfaces.

214712. Hilger Ltd., A., and another. POLARIZING APPARATUS.

The analyser has its dividing line at the focus of the eyepiece, the position being unaffected by introducing the substance under test. Light from the source is rendered parallel by a lens, and passed through a bichromate cell, a Nicol prism, and the tube of the instrument. At the eyepiece end are arranged two Nicol prisms and inclined plates, which are adjusted to maintain a sharp dividing line at the focus of the eyepiece. In use, the polarizer is rotated, and not the analyser.

215331*. Bünger, L. BINOCULARS.

Real images are formed by concave reflectors mounted at an acute angle to the optical axis, and are erected by prisms, of which one face is curved to correct for astigmatism. The optical elements are mounted in hinge frames of light metal. The reflectors are more widely separated than the eyepieces, and when in use lie at the sides of the head.

215337*. Optische Anstalt C. P. Goerz Akt-Ges. LENSES.

Orthoscopic eyepieces comprise a triple cemented component and a collective eye lens, together with a field lens separated from the cemented component by a very small air-space.

215536. Curry and Paxton Ltd., and another. OPHTHALMOSCOPES.

Ophthalmoscopes are fitted with a turn-table base so that either an illuminated or an unilluminated mirror can be employed. Holes in flexible strips on the base receive pins on the mirror plates, thus enabling various forms of mirror-holders to be fitted.

215673. Taylor, A., and another. TELESCOPES.

In a tacheometric telescope having a fixed object glass and measuring marks, the effect of a fixed, variable-power collective lens is secured by the use of a fixed collective component and a movable system comprising a collective and a dispersive lens mounted in a sliding tube.

* Specification published before acceptance.

215979. Chapman, W., and another. MICROSCOPE CONDENSERS.

Opaque objects are illuminated from above by light concentrated by an annular condenser, which surrounds the objective of the microscope.

215981. Taylor, A., and another. SURVEYING INSTRUMENTS.

The sighting-means for a miners' dial comprises a telescope of unit magnification consisting of two similar lenses mounted equidistant from a diaphragm or cross-wires, so that either may serve as the eyepiece. It is mounted in brackets upon the compass-box so that the inclination can be read off.

216518*. Thorner, W. OPTICAL SYSTEMS.

Multiple colour-images are simultaneously projected by an optical system consisting of a lens forming an enlarged image of the composite transparency, each portion of which is then projected upon the screen by its own system of lenses, each beam passing through the appropriate colour filter.

217262. Thorne-Baker, T. LENSES.

Achromatic objectives are constructed of materials transmitting a high proportion of ultra-violet light. Borosilicate crown may be combined with ultra-violet flint, or quartz with fluorite or Iceland spar.

R. J. T.

* Specification published before acceptance.